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GEOMETRY OF CONICS.

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THE

GEOMETRY OF CONICS.

SECOND EDITION, REVISED AND ENLARGED.

BY

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PREFACE TO THE FIRST EDITION.

THE work now in part published is the result of an attempt to reduce the chaos of Geometrical Conics to order. The subject having suffered not a little from desultory treatment, I have endeavoured to reconstruct it on a uniform plan, taking as a standard whereby to regulate the sequence of proofs the principle that *Chord-properties* should take precedence of *Tangent-properties*, the latter being deduced from the former rather than the former from the latter.

This principle commends itself from more than one point of view. In proofs of chord-properties it is superfluous to anticipate the comparatively complex notion of a tangent; while on the other hand the idea of a limit is the more clearly apprehended when approached in its natural order, and the properties of tangents are proved most convincingly when their intrinsic relation to the properties of chords is shewn. But, above all, this order refers the student consistently to first principles, thus rendering Geometry at once more thorough as a means of intellectual training, and more effective as an introduction to the powerful modern methods of Analysis.

Simplicity is of course essential in an elementary work; but I have attempted at the same time to secure comprehensiveness by assigning their due importance to properties by which the Conic is characterized in the Higher Geometry of Curves—such as the quadratic relation between its coordinates and the rectilinearity of its diameters. As regards the practical working of this arrangement, it is scarcely too much to say that in the Parabola the student who has thoroughly grasped Prop. II. has little more to learn. In the Rectangular Hyperbola the corresponding theorem leads by easy stages to the solution of all difficulties; while in the Ellipse and the general Hyperbola a single construction* suffices for the demonstration of all the leading properties of conjugate diameters.

Orthogonal Projection, already introduced under a new name†, will be further discussed in the sequel. The Right Cone will be treated somewhat less inadequately than heretofore; but, despite the skilful advocacy of Mr Stuart Jackson, I am unable to acquiesce in the primary definition of Conics from the solid. No allusion has yet been made to the Conjugate Hyperbola, which may be viewed as a contrivance for giving a false definiteness to the student's conceptions, and perpetuating his illusion that the Hyperbola is a discontinuous curve.

For the proof‖ of the chord-property that PN^2 varies as $AN.NA'$ my best thanks are due to Mr W. Allen Whitworth. The same proof was discovered independently by Mr Besant. In respect of the general tangent-property of Art. 108, I have pleasure in repeating my acknowledgments to Professor Adams. Mr Drew kindly places at my disposal his well-known proof that the tangent makes equal angles with the focal radii to its point of contact. In the Rectangular Hyperbola I have endeavoured to do justice to the investigations of Mr Wolstenholme, who has thrown much light upon that remarkable curve. I have profited by the advice and assistance of Mr Rawdon Levett of King Edward's School Birmingham, Secretary of the Association for the Improvement of Geometrical Teaching. It is needless to add that I am a debtor to Dr Salmon's inexhaustible works.

April, 1872.

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* The triangles pCN, dCR in Arts. 72, 212 differ only in relative position.

† See page 38.

‖ Arts. 53, 202.

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THE proofs in this work are so far independent that various propositions may be conveniently omitted in the first reading.

The order of the Chapters may also be varied. The Rectangular Hyperbola may with advantage be studied very early. Of all Conics it is that which as regards certain of its properties is most nearly identical with the circle; and the conception of its form is simplified by the use of the asymptotes as guiding lines.

The figures for the general Hyperbola are so arranged that they may be compared with the figures for the Ellipse.

THE GEOMETRY OF CONICS.

INTRODUCTION.

THE name CONIC is applied to a family of curves to which the circle belongs. It includes three varieties, the *Ellipse*, the *Parabola*, and the *Hyperbola*.

THE ELLIPSE

is the simplest in form. Its relation to the circle is shewn by the following construction, which will be found to lead to important results. From points in a circle let fall perpendiculars pN, dR,...to a fixed diameter AA'. If these perpendiculars be cut in a constant ratio the points of section will lie on an ellipse. [73.

By making the constant ratio of PN to pN very small we may flatten the ellipse indefinitely. And by making the ratio approximate to a ratio of equality we may make the ellipse as nearly circular as we please.

A construction much simpler in practice results from using two circles, thus: Let AA', BB' be diameters at right angles of two concentric circles. If from the points in which a common radius cuts the circles parallels be drawn to BB', AA' respectively, their intersection will describe an ellipse. [Ex. 130.

The following construction shews the relation of the ellipse to two points called the foci *. Let a loop of string, supposed inelastic, be passed over two pins at points S, S' (fig. p. 47), and let it be stretched into the form of a triangle, always in the same plane, by the point of a pencil or sharp instrument at P. Then as P moves in such a way as to keep the string stretched it describes an ellipse. It is evident that if S, S' be made to coincide the ellipse becomes circular, and if the distance SS' be made as great as possible, i.e. equal to half the length of the loop, the ellipse becomes flattened indefinitely. Example 98 gives another construction for the ellipse.

---

* The planets describe approximately ellipses having the sun in one focus. For this reason the first letter of *Sol* is used, as by Newton, to denote a focus.

THE PARABOLA

is the curve which would be described by a particle moving in a vacuum under the influence of gravity. A practical construction is given in Example 11.

THE HYPERBOLA

is most simply drawn by the analogous construction of Example 213.

THE CONE.

The three curves considered above were originally treated as plane sections of a Cone. Hence their old name *Conic Sections*. The cone and its sections may be shewn by means of a wooden model. An ellipse may also be cut from a cylinder or roller of circular transverse section. If the roller be cut obliquely, the section, supposed plane, will always be an ellipse. We may shew the sections optically by casting the shadow of a sphere or of a circular disc from a point of light upon a plane surface*. If the point of light be vertically under a point in the rim of the disc, the shadow thrown upon a vertical wall will be parabolic.

These constructions and illustrations will suffice to give a practical acquaintance at the outset with the forms of Conics. We shall not, however, in our fundamental definition make use of any of the constructions already given.

---

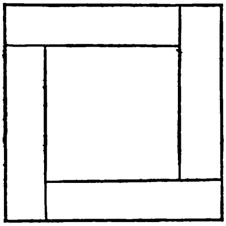
* They may also be shewn very roughly by means of a light held at the small end of a conical lamp-shade.

LEMMAS.

A. *To prove geometrically that*

$$(a+b)^2-(a\sim b)^2=4ab.$$

If four rectangles of sides a, b be fitted symmetrically about the square on $a\sim b$, the whole figure thus made up will be the square on $a+b$.



Therefore $\quad (a+b)^2=(a\sim b)^2+4ab$,

or $\quad (a+b)^2-(a\sim b)^2=4ab$.

The same is proved in Euc. II. 8, but by an unsymmetrical construction which shews only a gnomon of equal area instead of the four rectangles.

B. From the ends of a straight line QQ' and from its middle point v (fig. p. 6) let parallels QM, $Q'M'$, vO be drawn to meet any other straight line. Draw parallels from Q', Q to $M'M$, and let them meet vO in Z', Z.

Then in the triangles QvZ, $Q'vZ'$,

$$vZ=vZ',$$

or $\quad QM-Ov=Ov-Q'M'$.

Therefore $QM+Q'M'=2\,Ov$.

If the figure be drawn as on p. 66, where O bisects Qq, it may be shewn in like manner that

$$QM-qm=2OL.$$

If the parallels drawn as above be called the ordinates of the points from which they are drawn, we may enunciate the Lemma as follows:

The ordinate of the centre of a straight line is equal to half the sum or difference of the ordinates of its ends.

As a particular case of the above, if S be a point in the straight line itself, then

$$SQ \pm SQ' = 2Sv.$$

For, $SQ - Sv = \frac{1}{2}QQ' = Sv + SQ'$; [Fig. p. 6.
and similarly when S cuts QQ' externally.

C. In the figure of Art. 78, let PN be drawn perpendicular to SS'. Then by Euc. II. 12, 13,

$$SP^2 = CS^2 + CP^2 + 2CS \cdot CN,$$

and $$S'P^2 = CS'^2 + CP^2 - 2CS' \cdot CN.$$

By addition, if C bisects SS',

$$P^2 + S'P^2 = 2CP^2 + 2CS^2.$$

D. In Art. 96, let a parallel from O to RS meet SP in M.

Then $SM : OR = SP : PR.$

And because the bisector of the angle PSQ is perpendicular to OM, and is nearer to SO than to SM; therefore SO is less than SM.

Therefore $SO : OR < SP : PR.$

DEFINITION.

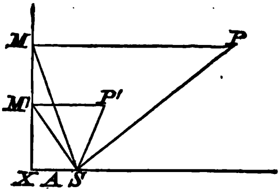
The term *Equivalent* will be used in this work to denote *equal and similar* or *equal in all respects.*

CHAPTER I.

CHORD-PROPERTIES OF THE PARABOLA.

DEF. A parabola is the curve described in a plane by a point which moves in such a way that its distance from a certain fixed point, called the *Focus*, is always equal to its perpendicular distance from a certain fixed straight line, called the *Directrix*.

1. *With any straight line* MX *as directrix, and any point* S *exterior to it as focus, a parabola may be described.*

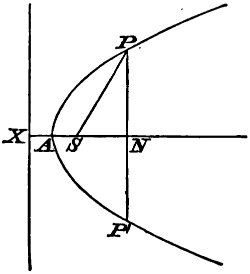


From any point M on the directrix draw a straight line at right angles to the directrix. On the straight line so drawn one point P belonging to the parabola can be found; for if the angle MSP be made equal to the known angle MSX, or SMP, then $SP=PM$, or P is a point on the curve. In this way any number of points P, P' belonging to the parabola may be found; and they all lie on the same side of the directrix with the focus.

The straight line through the focus at right angles to the directrix is called the *Axis*. The point in which the axis meets the parabola is called the *Vertex*. Let the axis meet the directrix in X. The vertex A is the centre of the straight line SX, since from the definition the distance of A from S must be equal to its perpendicular distance AX from the directrix.

2. From the construction already given it appears that on every straight line PM perpendicular to the directrix, or parallel to the axis, one point and one only belonging to the curve can be found. In other words, if a straight line parallel to the axis cuts the parabola in a point P it *cannot cut it again.* Hence the parabola is an open curve; and it spreads out to infinity on both sides of the axis, since the point M, and therefore also P, may be taken as far as we please from the axis and on either side of it. The curve recedes at once from the directrix and the axis, since the angle PSX, or $2MSX$, increases with MX.

DEF. The perpendicular PN let fall upon the axis from any point P is said to be the *Principal Ordinate*, or briefly the *Ordinate* of P; and AN is said to be the *Principal Abscissa*, or briefly the *Abscissa* of P.



Since $NX = PM$ = *the perpendicular distance of* P *from the directrix*, therefore if P be a point on the parabola, $SP = NX$. It is sometimes convenient to express the definition in this form.

3. PROP. I. *If* PN, AN *be the principal ordinate and abscissa of any point* P *on the parabola, then*

$$PN^2 = 4AS.AN.$$

From the definition $SP^2 = NX^2$.

Hence

$$(AN \sim AS)^2 + PN^2 = (AN + AS)^2 \qquad \text{[Euc. I. 47.}$$

$$= 4AS.AN + (AN \sim AS)^2. \qquad \text{[Introd.}$$

Therefore $\qquad PN^2 = 4AS.AN.$

DEF. A straight line joining any two points on a curve is said to be a *Chord*. A chord which passes through the focus is said to be a *Focal Chord*. The focal chord perpendicular to the axis is called the *Latus Rectum*.

4. COR. The square of the focal ordinate or semi-latus rectum is equal to $4AS.AS$. Hence the latus rectum is equal to $4AS$. Hence also, the ordinate is a mean proportional between the abscissa and the latus rectum.

5. *Conversely, if* P *be such that* $PN^2 = 4AS.AN$, *where the length* AS *is given, or in other words, if* PN^2 *vary as* AN, *then will the locus of* P *be a parabola.*

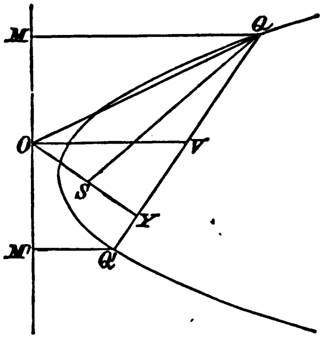
The parabola may be very simply described with the help of Prop. I. To find successive points on the curve, we may proceed as follows. Measure off an abscissa AN of any length, for example, of length $3AS$. Then by the proposition, $PN^2 = 4AS.3AS$. Therefore $PN = 2\sqrt{3}AS$. Now through the point N draw a straight line perpendicular to the axis, and on the line thus drawn take a point P at distance $2\sqrt{3}AS$ from the axis. Then P is a point on the parabola. Thus by measuring off abscissæ of different lengths and using Prop. I. to find the lengths of the corresponding ordinates, we may determine successively any number of points on the parabola. In this construction the ordinates may be measured either upwards or downwards from the axis. Thus points on the curve are found in pairs as P, P' which have a common abscissa and equal ordinates. Hence the parabola is said to be *symmetrical with respect to its axis*, as may be seen likewise from Art. 1.

EXAMPLES.

1. If the ordinates of a parabola be cut in a constant ratio the points of section will lie on a parabola.

2. Circles whose radii are in arithmetical progression touch a given straight line on the same side at a given point. If to each circle a tangent parallel to the given line be drawn, it will cut the circle next larger in points lying on a parabola.

6. PROP. II. *The locus of the middle points of any system of parallel chords of a parabola is a straight line parallel to the axis. And the bisecting line meets the directrix on the straight line through the focus perpendicular to the chords.*



Take QQ', any one of a system of parallel chords. Let the focal perpendicular on the chords meet QQ' in Y, and the directrix in O. Let fall perpendiculars QM, $Q'M'$ on the directrix. Then shall $OM = OM'$.

For $OM^2 + QM^2 = OQ^2$ [Euc. I. 47.

$= SQ^2 + SO^2 + 2OS.SY.$ [Euc. II. 12.

And $QM^2 = SQ^2$, from the definition.

By subtraction $OM^2 = SO^2 + 2OS.SY.$

Similarly $OM'^2 = SO^2 + 2OS.SY.$

Hence $OM = OM'$, and the straight line through O parallel to the axis bisects QQ'; that is to say, it bisects every chord to which OS is perpendicular.

7. Otherwise, by Euc. I. 47: Since the squares of OY, QY, and those of SY, QY are equal respectively to OQ^2, and to SQ^2, therefore

$$OY^2 - SY^2 = OQ^2 - SQ^2 = OQ^2 - QM^2 \quad \text{[Def.}$$
$$= OM^2.$$

And OM'^2 has the same value.

Def. A straight line which bisects a system of parallel chords is called a *Diameter*.

8. In the parabola, all diameters are parallel to the axis. Conversely every straight line parallel to the axis is a diameter. It is evident that a diameter is determined when the centres of any two of its bisected chords are given. The axis may be called the *Principal Diameter*.

9. It is important that the student should familiarize himself with Prop. II. It may be well to state it in converse forms thus:

(i) If QQ' be any chord, and if the perpendicular upon it from the focus meet the directrix in O, then if OV be drawn to the centre of the chord it is parallel to the axis; or if it be drawn parallel to the axis it bisects the chord.

(ii) If the diameter through the centre V of any chord meet the directrix in O, and if OS meet the chord in Y, then OYV is a right angle.

In the particular case of a *focal chord* bisected in v (fig. Prop. III.) $\angle OSv =$ a right angle; and conversely if OSQ be a right angle, the diameter through O bisects QQ'.

Examples.

3. Using the method and notation of Art. 159, shew that for the parabola

$$OL \,.\, QK = SX \,.\, qK.$$

Apply this result to prove Prop. II.

4. If from the point O in which a diameter meets the directrix OZ be drawn parallel to the bisected chords and meeting the axis in Z, then

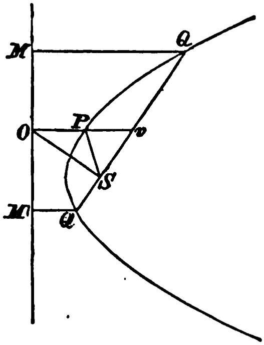
$$OX^2 = SX \,.\, XZ.$$

5. The circle on a chord of a parabola as diameter does not meet the directrix unless the chord passes through the focus.

6. Shew that $\angle QOQ' = MYM'$. [6.

7. The perpendicular to a chord of a parabola from its middle point V meets the axis at a distance equal to SX from the foot of the ordinate of V.

10. PROP. III. *To find the length of any focal chord.*



Take the point O on the directrix such that $\angle OSQ =$ a right angle. Let a straight line through O parallel to the axis meet the curve in P and the chord in v. Then v is the middle point of QQ'*. [9.

Let fall perpendiculars QM, $Q'M'$ upon the directrix.

Then $SQ = QM$, and $SQ' = Q'M'$. [Def.

By addition, $QQ' = QM + Q'M' = 2Ov$, since v is the centre of QQ'. [Introd.

But because P lies on the curve, $SP = PO$. Therefore the angles PSO, POS are equal, and their complements PSv, PvS are equal. Therefore $Pv = PS = PO$, or $Ov = 2SP$.

Therefore $QQ' = 2\,Ov = 4SP$.

11. We shall now give more general definitions of the terms *Ordinate* and *Abscissa*.

Let a diameter meet the curve in P, and meet QQ', any one of the chords which it bisects, in V. Then QV is the *Ordinate* and PV is the *Abscissa* of Q.

More generally, if Q be any point whatever in the plane of the parabola, and QV be measured parallel to the chords bisected by PV, then QV, PV are the ordinate and abscissa of Q referred to the diameter through P.

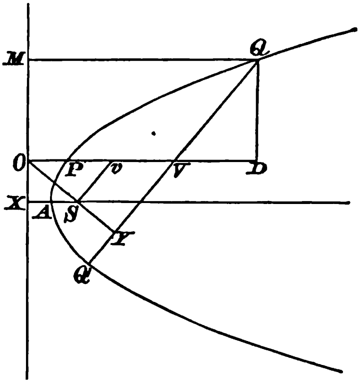
The ordinates of all points on the curve referred to any fixed diameter are parallel. [6.

---

* Or thus, without assuming Prop. II.: In the right-angled triangles MQO, SQO, which have MQ, $QO = SQ$, QO, the side $OM = OS = OM'$, similarly. Hence $Qv = Q'v$.

12. Prop. IV. *If* PV *be the abscissa of any point* Q *on the parabola, then*

$$QV^2 = 4SP \, . \, PV.$$



Let SY meet a chord QQ' perpendicularly in Y, and meet the directrix in O. Draw a parallel to the axis through O meeting the curve, $Q'Q$, and the parallel focal chord, in P, V, v. Let fall perpendiculars QD, QM on OV and the directrix.

Then $OY : OV = SY : vV$, by parallels.

Hence $OY^2 - SY^2 : OV^2 - vV^2 = OY^2 : OV^2$,

$= QD^2 : QV^2$,

by similar right-angled triangles OYV, QDV.

But $QD^2 = OM^2 = OY^2 - SY^2$, as in Art. 7.

Therefore $QV^2 = OV^2 - vV^2$.

Also $Pv = SP = PO$, as in Art. 10.

Therefore $OV = PV + SP$,

and $vV = PV \sim SP$.

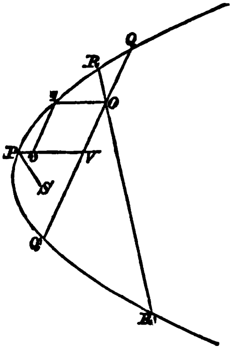
Therefore $OV^2 - vV^2 = 4SP \, . \, PV$. [Introd.

Hence $QV^2 = 4SP \, . \, PV$, from above.

And, $Q'V^2$ having the same value, QV is a semi-chord or ordinate.

13. **Cor.** The ordinate QV is a mean proportional between the abscissa PV and the parallel focal chord. [10.

14. **Prop. V.** *The rectangles contained by the segments of any two intersecting chords are to one another as the lengths of the parallel focal chords.*



Let the diameters through O the intersection of any two chords QQ', RR', and through V the middle point of QQ', meet the curve in q, P respectively. Let Pv be the abscissa of q.

$$\begin{aligned} \text{Then} \qquad QO \, . \, OQ' &= QV^2 - OV^2 && \text{[Euc. II. 5, Cor.} \\ &= QV^2 - qv^2 \\ &= 4SP \, . \, PV - 4SP \, . \, Pv. && \text{[12.} \end{aligned}$$

But $PV - Pv$ is equal to vV or qO.

Therefore $\qquad QO \, . \, OQ' = 4SP \, . \, qO.$

Similarly, if P' be the point in which the diameter bisecting RR' meets the curve,

$$RO \, . \, OR' = 4SP' \, . \, qO.$$

Hence $\quad QO \, . \, OQ' : RO \, . \, OR' = 4SP : 4SP'.$

15. **Cor.** The ratio of these rectangles *is independent of the position of* O*. It is the same for every pair of chords parallel to QQ', RR' respectively, since for all such chords P and P' remain unaltered. [6.

---

* This point may be either external to the parabola or internal.

8. A point within a parabola is nearer to the focus than to the directrix.

9. All parabolas are similar curves. Prove this by deducing from the bisection of the angle PSX by SM that the parallel focal radii of similarly situated parabolas are proportional. [1.

10. If the focal radii of a parabola be cut in a constant ratio the points of section will lie on a parabola.

11. Prove the following construction. Take any ordinate NP, and draw PM parallel and equal to NA. Divide NP into any number of equal parts, and through the points of section draw parallels p_1, p_2, p_3,...to the axis. Divide MP into the same number of equal parts in points 1, 2, 3,... Then the lines p_1, p_2, p_3,...meet $A1$, $A2$, $A3$,... respectively on the parabola.

12. Shew that $\angle QOQ' =$ a right angle $= MSM'$. [10.

13. The semi-latus rectum is a mean proportional between the principal ordinates of the ends of a focal chord. And if AM, AM' be the corresponding abscissæ, then $AM \,.\, AM' = AS^2$.

14. The circle on a focal chord touches the directrix.

15. Shew that

$$SO^2 = SX \,.\, Ov = 4AS \,.\, SP. \qquad [12.$$

16. Also that

$$QD^2 = 4AS \,.\, PV. \qquad [12.$$

17. Deduce from Ex. 9, that the ratio of the rectangles in Art. 15 is the same for all similarly situated parabolas.

18. A circle can be described touching any two diameters of a parabola and the focal radii to their extremities.

19. The locus of the centre V of a focal chord is another parabola. Prove this by shewing that if VL be the ordinate of V, then

$$VL^2 = 2AS \,.\, AL. \qquad [\text{Ex. } 4.$$

20. A chord QQ' is cut in O by a diameter which meets the curve in P. Shew that if R be a point on the curve whose abscissa is PO, and PV, PV' be the abscissæ of Q, Q', then

$$QV^2 - Q'V'^2 : QV^2 - OR^2 = QV + Q'V' : QV. \qquad [12.$$

Deduce that OR is a mean proportional between QV, $Q'V'$.

21. Shew also that $PV \cdot PV' = PO^2$.

22. Shew how to place in a parabola a focal chord of given length.

23. A parabola being given find its axis and focus.

24. If PQ be a chord which subtends a right angle at A, and AN, AM be the principal abscissæ of P, Q, then PQ passes through a fixed point, and

$$AN \cdot AM = PN \cdot PM = 16AS^2.$$

25. If PQ be a focal chord, AP, AQ meet the latus rectum at distances from S equal to the ordinates Q, P. [Ex. 13.

26. If the diameter at P meets the latus rectum in L, the intersection of AL, XP is on the curve.

27. A point on a parabola being given, then if the focus be given the envelope of the directrix is a circle; or if the directrix be given the locus of the focus is a circle.

28. Given the directrix of a parabola and two points on the curve; shew that two positions of the focus can be determined.

29. Given the focus of a parabola and two points on the curve; shew that two positions of the directrix may be found.

30. If two parabolas have a common focus, their common chord passes through the intersection of their directrices and bisects the angle between them.

31. If two parabolas have a common directrix, their common chord bisects the straight line joining their foci at right angles.

32. The straight lines which join the ends of a focal chord to the vertex meet the directrix on the diameters through the ends of the chord.

33. Find the locus of the centre of a circle which passes through a given point and touches a given straight line.

34. A point moves in such a way that its perpendicular distance from a fixed straight line is less by a constant quantity than its distance from a fixed point. Shew that the point describes a parabola.

35. Find the locus of the centre of a circle which touches a given circle and a given straight line.

36. Circles being described on the segments of a focal chord as diameters, the straight line joining their centres subtends right angles at the intersections of their common tangents.

37. The perpendicular from Q to a chord AQ meets the axis at a distance equal to the latus rectum from the foot of the ordinate of Q.

38. The straight lines joining any point on a parabola to the ends of a focal chord intercept on the directrix a length which subtends a right angle at the focus.

39. Parallel chords being cut by any diameter, the rectangles contained by their segments are as the abscissæ of the points of section.

40. Equal focal chords of a parabola are equally inclined to the axis. Hence shew that the common chords of a circle and a parabola are equally inclined to the axis. [14.

41. A circle cuts a parabola in three points, whether on the same side of the axis or on different sides. Prove that it cuts the curve in one other point and one only, and shew how to find this point.

42. If a circle cuts a parabola in points 1, 2, 3 above the axis and in a point 4 below it, the difference of the ordinates of 1, 3 is to the difference of their abscissæ as the sum of the ordinates of 2, 4 to the difference of their abscissæ. Deduce that the ordinate of 4 is equal to the sum of the ordinates of 1, 2, 3.

43. If 1, 2 and 3, 4 lie on opposite sides of the axis the sum of the ordinates of 1, 2 is equal to the sum of the ordinates of 3, 4.

44. On a chord through a fixed point O there is taken a mean proportional OM to the segments of the chord. Shew that the locus of M is a diameter.

CHAPTER II.

TANGENT-PROPERTIES OF THE PARABOLA.

16. IF the extremities of a chord be made to approach one another continuously the chord tends to assume a certain *limiting* position; and in this limiting position, viz. when its extremities have become coincident, the chord produced indefinitely has become a *Tangent.* Thus in fig. p. 14 suppose Q' to move along the curve up to Q. Then ultimately QQ' becomes the tangent at Q.

There are various ways in which a chord may be made to assume a position of tangency. It is often convenient to use the following.

17. In fig. p. 14 Qq is a chord bisected by the diameter TV. Suppose this chord to move *parallel to itself* into a new position $Q'q'$. It is still bisected by TV, say in V'. [6.

Suppose it now to move continuously in the same way until its middle point reaches the end of the diameter TV; then each of the portions QV, qV, which have been diminishing continuously, will have vanished, and the two points in which the chord met the curve will have been brought into coincidence; that is to say the chord will have become a tangent, viz. at the end P of the diameter which bisects Qq. Hence

The tangent at the extremity of any diameter is parallel to the ordinates of that diameter.

In particular

The tangent at the vertex is parallel to the principal ordinates, or perpendicular to the axis.

18. To take another example:—In Art. 14 we have shewn that the rectangles contained by the segments of any two intersecting chords are to one another as the lengths of the parallel focal chords. This holds whether the point of intersection be internal or external to the curve. Let it be external as T in fig. p. 14.

Then $TQ.TQ'$ is to $Tq.Tq'$ in the ratio of the focal chords to which QQ', qq' are parallel.

Let QQ' move parallel to itself until it becomes the tangent at some point Q_1. Then $TQ.TQ'$ becomes TQ_1^2.

In like manner let qq' become the tangent at some point q_1. Then $Tq.Tq'$ becomes Tq_1^2.

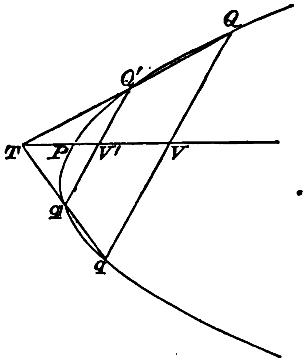
Therefore the squares of any two tangents TQ_1, Tq_1 are to one another in the ratio of the focal chords to which they are parallel.

19. Again (9), if a diameter meet the directrix in O, then every chord QQ' bisected by that diameter is perpendicular to OS. Therefore in the limit *the tangent at* P (fig. p. 7) *the extremity of the diameter* OV *is perpendicular to* OS, *or*, in fig. p. 16, *to* MS. And conversely, *if the focal perpendicular on the tangent at* P *meets the directrix in* M, *then* PM *is a diameter*, and is therefore parallel to the axis or perpendicular to the directrix.

20. We might also, after the method of Euclid, regard a tangent as a straight line which *meets* a conic *and being produced does not cut it*, or which, except at the point of contact, lies wholly on the convex or outer side of the conic. We shall briefly indicate the method to be pursued when the Euclid definition is adopted.

From any point P on the parabola (fig. p. 16) let fall perpendiculars PM, PY on the directrix and SM respectively. Then will PY be the tangent at P. For since evidently Y is the centre of SM, therefore if t be any point other than P on PY, then $St = TM$. Therefore St *is greater than the perpendicular distance of* t *from the directrix*. Hence it may be shewn that every point t on PY lies on the convex side of the parabola, or PY is the tangent at P. [51.

21. PROP. VI. *Tangents at the extremities of any chord intersect on the diameter which bisects the chord.*



Let Q, Q' and q, q' be adjacent extremities of any two parallel chords, and let QQ', qq' meet in T, then shall T lie on the diameter which bisects the chords.

For let TV, drawn to the centre of Qq, cut $Q'q'$ in V'. Then by parallels,

$$Q'V' : q'V' = QV : qV.$$

But $QV = qV$. Therefore $Q'V' = qV'$.

Hence TV, since it bisects both Qq and $Q'q'$, is the diameter which bisects chords parallel to Qq. [8.

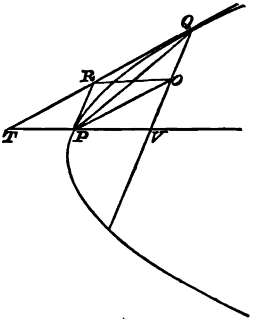
Now as $Q'q'$ moves parallel to itself up to Qq, the point of intersection T always lies on the diameter through V. And this being true always, is true in the limit, viz. when QQ', qq' become the tangents at Q, q.

Hence the tangents at the ends of the chord Qq meet on the diameter which bisects the chord.

22. Conversely, if the tangents at P, Q meet in R, the straight line drawn through R to bisect the chord of contact PQ is a diameter.

23. PROP. VII. *If* PV *be the abscissa of any point* Q *on the parabola, and if the tangent at* Q *meet the diameter* PV *in* T, *then*

$$PV = PT.$$

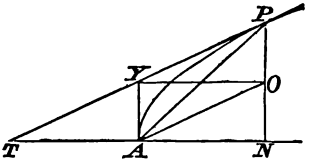


Let the tangent at P, which is parallel to the ordinate QV, meet QT in R. Complete the parallelogram $QRPO$, viz. by drawing PO parallel to RQ. Then the diagonal RO bisects the diagonal PQ. That is to say, RO bisects the chord of contact of the tangents RP, RQ. Therefore RO is a diameter (22), and is parallel to any other diameter, as PV, since all diameters are parallel to the axis.

Hence by parallels,

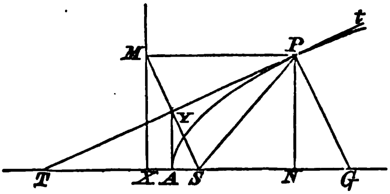
$$PV = RO = PT.$$

24. The following particular case is to be noticed.



If PN be the principal ordinate of P, then $AN = AT$.

25. PROP. VIII. *The tangent at* P *bisects the angle between* SP *and the diameter through* P.



Draw PM perpendicular to the directrix, and let SM meet the tangent at P in Y.

Then in the triangles SYP, MYP, since

$$SP, PY = MP, PY,$$

each to each, and the angles at Y are right angles, [19.

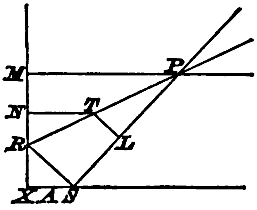
therefore $\angle SPY = MPY.$

26. COR. If the tangent meet the directrix in R, it readily follows that $\angle RSP = RMP =$ a right angle. But this is best proved independently as in Art. 105.

27. COR. If the tangent meet the axis in T, then by parallels and by the proposition, $\angle STP = MPT = SPT$.

28. PROP. IX. *If from any point* T *on the tangent at* P *perpendiculars* TL, TN *be let fall on* SP *and the directrix, then*

SL = TN.



If the tangent meet the directrix in R the angle RSP is a right angle. [26.

Therefore TL, RS are parallel. So too are TN, PM.

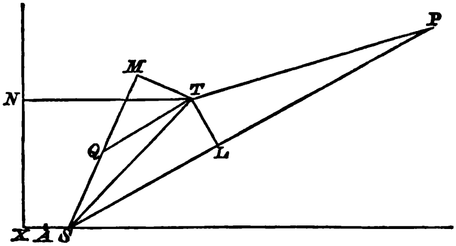
Hence $SL : SP = TR : PR$ [Euc. VI. 2.

$= TN : PM$.

But $SP = PM$.

Therefore $SL = TN$.

29. Prop. X. *If the tangents at* P, Q *meet in* T, TP, TQ *subtend equal angles at* S.



On SP, SQ and the directrix let fall perpendiculars TL, TM, TN.

Now because T is a point on the tangent at P, therefore $SL = TN$. [28.

And because T is a point on the tangent at Q, therefore $SM = TN$. [28.

Hence SL, SM are equal. Therefore in the right-angled triangles STL, STM the angles at S are equal; that is to say, TP, TQ subtend equal angles at S.

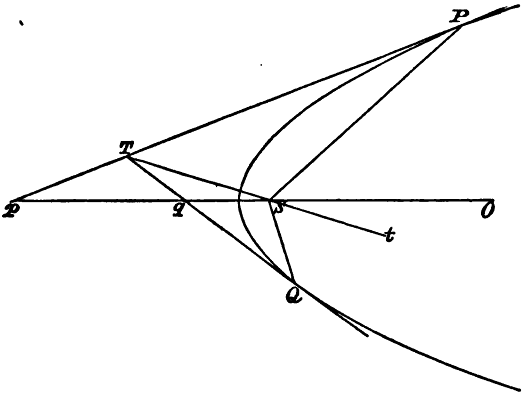
Examples.

45. The portion of any tangent intercepted by the tangents at the ends of a parallel chord is bisected at the point of contact.

46. Apply this result to determine the length of a focal chord.

47. Any length measured on a tangent subtends equal angles at the focus and at the point in which the diameter through the point of contact meets the directrix.

30. PROP. XI. *The acute angle between any two tangents is equal to half the angle which their chord of contact subtends at the focus.*



Let the tangents at P, Q intersect in T and meet the axis in p, q. Take any point O in pS produced.

Then by Euc. I. 32,

$$\angle PSO = SPp + SpP = 2SpP. \qquad [27.$$

Similarly, $\angle QSO = \qquad\qquad = 2SqQ.$

Hence:

(i) If P, Q lie on opposite sides of the axis, then by addition,

$$\angle PSQ = 2(SpP + SqQ) = 2(SpP + pqT)$$
$$= 2PTQ. \qquad \text{[Euc. I. 32.}$$

(ii) If P, Q lie on the same side of the axis*, then by subtraction,

$$\angle PSQ = 2(SqQ - SpP) = 2pTq.$$

Therefore in each case the acute angle between the tangents is equal to half the angle PSQ.

31. PROP. XII. *If the tangents at* P, Q *meet in* T, *the triangles* SPT, STQ *are similar.*

* Use the figure on p. 17, supplying the letters p, q, O.

If P, Q lie on opposite sides of the vertex, then in the preceding figure produce TS to t.

Then $\angle PSt = \frac{1}{2} PSQ$ [29.

$= PTQ.$ [30.

Hence by Euc. I. 32,

$$\angle STP + SPT = STQ + STP.$$

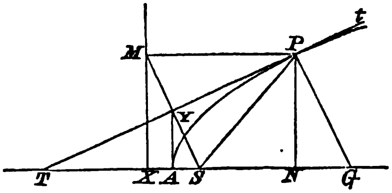
Therefore $\angle SPT = STQ.$

And $\angle TSP = TSQ.$ [29.

Therefore the triangles SPT, STQ are similar.

32. Prop. XIII. *If* SY *be the focal perpendicular on the tangent at* P, *then* Y *lies on the tangent at* A, *and*

$$SY^2 = SA \cdot SP.$$



Let SY meet the directrix in M.

Then PM is a diameter (19), and is therefore perpendicular to the directrix.

Hence in the right-angled triangles SPY, MPY,

$$SP = PM.$$

Also PY is common. Therefore $SY = MY$.

Hence AY, since it bisects both SM and SX, is parallel to MX, and *is therefore the tangent at* A. [17.

And since, from what precedes, the triangles SPY, MPY are similar, therefore

$$\angle PSY = PMY = YSA, \text{ by parallels.}$$

Hence by similar right-angled triangles SPY, SYA,

$$SY : SP = SA : SY,$$

or

$$SY^2 = SA \cdot SP.$$

33. **Cor.** Any two tangents TP, TP' are as SY, SY' the focal perpendiculars upon them.

For $$TP^2 : TP'^2 = SP : SP' \qquad [18.$$
$$= SY^2 : SY'^2.$$

34. **Def.** The *Normal* at any point of a curve is the straight line drawn through that point at right angles to the tangent.

The *Subtangent* is the portion of the axis intercepted between the tangent and the principal ordinate.

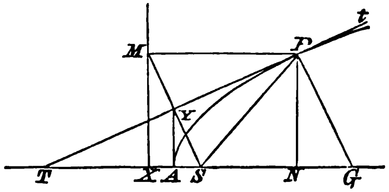
The *Subnormal* is the portion of the axis intercepted between the normal and the principal ordinate.

35. **Prop.* XIV.** *The subtangent is double of the principal abscissa; and the subnormal is equal to half the latus rectum.*

(i) Prove as in Prop. VII. that $AN = AT$.

Hence the subtangent $NT = 2AN$.

(ii) Let the normal at P meet the axis in G; and let the diameter through P meet the directrix in M. Then PG, being at right angles to the tangent, is parallel to MS. [19.



Hence the triangles PNG, MXS are similar; and, being also of equal altitude, they are equal in all respects.

Therefore the subnormal $NG = SX = 2AS$.

36. Or thus: Because TPG is a right angle, therefore
$$PN^2 = NG . NT$$
$$= NG . 2AN, \text{ from above.}$$
But $$PN^2 = 2AS . 2AN. \qquad [3.$$
Therefore the subnormal $NG = 2AS$.

37. It should be noticed that

$$SG = SP = ST. \qquad [27.$$

38. In the figure on p. 6, since OS is at right angles to QQ', therefore OQ, OQ' are the tangents at Q, Q'. [26.

And, since the right-angled triangles OSQ, OMQ have their sides SQ, QM equal, and QO common, therefore

$$\angle QOS = QOM$$
$$= \tfrac{1}{2} MOS.$$

Similarly $\angle Q'OS = \tfrac{1}{2} M'OS.$

By addition $\angle QOQ' =$ a right angle.

Hence in the parabola tangents at the ends of a focal chord meet at right angles on the directrix; and conversely, the directrix is the locus of intersection of tangents at right angles.

39. It will be seen that some of the definitions in Chapters I. II. are applicable to all conics; as, for example, of *Axis*, *Vertex*, *Ordinate*, *Latus Rectum*, *Diameter*, *Tangent*, *Normal*, *Subtangent*, *Subnormal*. So too is Art. 21.

EXAMPLES.

48. Shew how to draw a tangent to a parabola from a point on the tangent at the vertex.

49. If a leaf of a book be folded so that one corner moves along an opposite side the direction of the crease touches a parabola.

50. Shew how to draw tangents to a parabola from any given external point.

51. If PQ be the chord of contact of tangents from O, then $SO^2 = SP \cdot SQ$.

52. The tangent at Q meets the tangent at P in R and the diameter through P in T. If PT meets the directrix in M, shew that the triangles MPR, RPT are similar, so that $PR^2 = PM \cdot PT$.

53. Hence shew that, PV being the abscissa of Q,

$$QV^2 = 4SP \cdot PV.$$

54. Any two tangents TP, TQ make equal angles with TS and the diameter through T respectively.

55. If from any point T on a fixed tangent a second tangent TP be drawn, the angle STP will be constant.

56. The vertices of any circumscribed triangle are concyclic with the focus.

57. PQR being a circumscribed triangle, the perpendiculars from P, Q, R to SP, SQ, SR cointersect.

58. If the tangents at P, Q meet in T, and if C be the centre of the circle TPQ, then CST is a right angle.

59. The normal at P bisects the angle between SP and the diameter through P.

60. Two parabolas which have a common focus and their axes in opposite directions intersect at right angles.

61. The perpendicular drawn to a normal from the point in which it meets the axis envelopes an equal parabola.

62. Normals at the extremities of a focal chord intersect on the diameter which bisects the chord.

63. If PQ be a focal chord, and R the foot of the perpendicular upon it from the intersection of the normals at P, Q, then $SP = QR$.

64. The portion of any tangent intercepted by the tangents at fixed points P, Q subtends a constant angle at S. The angle subtended is a right angle when PQ passes through S.

65. If a circle through S touches the parabola in P, Q, then SP is equal to the latus rectum.

66. The normal at any point is equal to twice the focal perpendicular upon the tangent, and is also a mean proportional between the focal distance of that point and the latus rectum.

67. The squares of the normals at the ends of a focal chord are together equal to the square of twice the normal perpendicular to the chord.

68. The locus of the vertex of a parabola which has a given focus and touches a given straight line is a circle.

69. The circle on a focal radius touches the tangent at A.

70. The tangent and normal at any point are bisected by the focal perpendiculars upon them; and the straight line joining the feet of the perpendiculars is parallel to the axis.

71. The diameter through one end of a focal chord bisects the chord normal at the other.

72. The locus of the foot of the focal perpendicular on the normal is a parabola.

73. If the tangents at P, Q intersect in R, the circle through P touching QR in R passes through S.

74. If from the foot of the normal at P a perpendicular GK be drawn to SP, then $PK = 2AS$.

75. Determine the position of P so that the triangle SPG may be equilateral.

76. The tangent at any point meets the directrix and the latus rectum in points equidistant from the focus.

77. If QQ' be the focal chord perpendicular to the normal at P, then $PG^2 = SQ \, . \, SQ'$.

78. The triangle bounded by three tangents to a parabola is equal to half the triangle whose vertices are at the points of contact.

79. If two parabolas be described each touching two sides of a given equilateral triangle at the points in which it meets the third side, prove that they have a common focus and that the tangent to either of them at their point of intersection is parallel to the axis of the other.

80. Two equal parabolas have the same axis and directrix. From a point on one of them tangents are drawn to the other. Shew that the perpendicular from that point to the chord of contact is bisected by the axis.

81. Supposing the triangle 123 in Ex. 42 to become evanescent, shew that in the limit the common chord of the circle and the parabola is equal to four times their common tangent measured from the curve to the axis.

82. A diameter meeting a chord and the tangent at an end of it is cut by the curve in the ratio in which it cuts the chord.

83. Draw a chord which shall be cut in a given ratio by a given diameter.

84. A parabola being inscribed in a triangle its directrix passes through the orthocentre.

CHAPTER III.

CHORD-PROPERTIES OF THE ELLIPSE.

40. DEF. A Conic is the curve described in a plane by a point which moves in such a way that its distance from a certain fixed point, called the *Focus*, is in a constant ratio to its perpendicular distance from a certain fixed straight line, called the *Directrix*.

This constant ratio is called the *Eccentricity*.

A Conic is called an *Ellipse*, a *Parabola*, or a *Hyperbola*, according as its eccentricity is less than, equal to, or greater than unity.

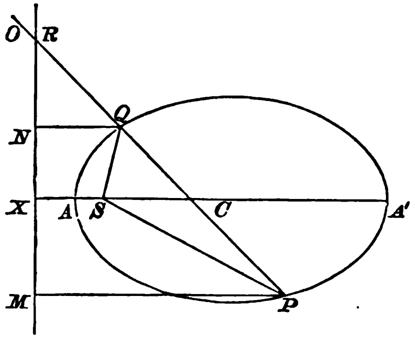
41. Some properties can be proved as simply for all conics at once as for the ellipse separately. We shall accordingly, in enunciating some of the propositions in Chapters III. and IV., use or imply the general term *Conic* instead of *Ellipse*. Articles which apply to all conics will be distinguished by the mark ¶.

42. Let S be the focus of an ellipse, and X the point in which the axis meets the directrix. Divide SX in A so that SA may be to AX as the eccentricity. Then A is a vertex. Since the eccentricity is less than unity it is evident that there is a second vertex A' in XS produced, such that SA' is to $A'X$ as the eccentricity.

43. The ellipse lies wholly on the same side of the directrix with the focus. For imagine two of its points O, P to lie on opposite sides. Let OP cut the directrix in R. Then,

by the definition if OP be at right angles to the directrix, and otherwise *a fortiori*,

$$SO < OR, \text{ and } SP < PR.$$



By addition, $SO + SP < OP$, which is impossible.

Hence the ellipse lies wholly on one side of the directrix, viz. that on which are the points A, A' already determined.

¶ 44. *An extension of the definition of a conic.*

From any point P on a conic draw PM perpendicular to the directrix, and PR meeting the directrix at *any constant angle*. Then $PM : PR$ is a constant ratio. But by definition $SP : PM$ is constant. Therefore $SP : PR$ is constant. Hence a conic might have been defined as the locus of a point P whose distance SP from the focus is in a constant ratio to its distance PR from the directrix measured parallel to any fixed straight line which meets the directrix. When this fixed straight line meets the directrix perpendicularly we come back to the original definition.

¶ 45. As a particular case of the above let P, Q be two points on a conic*, and let the straight line joining them meet the directrix in R. Let fall perpendiculars PM, QN on the directrix.

Then $$SP : SQ = PM : QN \qquad \text{[Def.}$$
$$= PR : QR,$$

by parallels, and conversely if this relation holds and P be on the curve, then Q will be on the curve.

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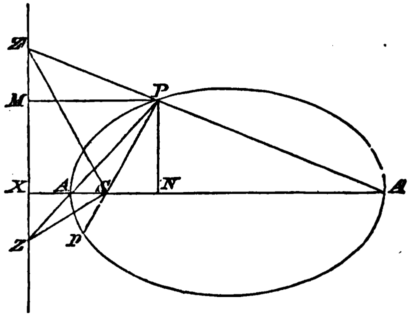
* This includes the case of a focal chord.

46. For two chords of an ellipse through P and A, A' respectively which meet the directrix in Z, Z',



$SA : SP = AX : PM$ [Def.

$= AZ : PZ$, by parallels;

and $SA' : SP = A'Z' : PZ'$, similarly.

¶47. From Art. 45, when one point P on the curve is given a second Q may be determined by drawing PR to any point R on the directrix, and then drawing SQ making the same angle as PS with SR. For SQ will meet RP in a point Q such that

$SP : SQ = PR : QR$. [Euc. VI. A.

From this construction it appears that a straight line which meets a conic in a point P will in general meet it in one other point Q, and that no straight line can meet a conic in more points than two. Hence Conics are called *curves of the second degree.*

48. Starting from the vertex A we may determine any number of points as P on the ellipse, by drawing AZ to any point Z on the directrix, and then making the angle ZSp equal to the known angle XSZ. The line pS thus drawn meets ZA on the ellipse. [47.

Or we might have determined points as P by starting from the vertex A' and taking the angle $Z'SP$ equal to XSZ', where Z' is any assumed point on the directrix.

49. *The straight lines joining the vertices to any point* P *on the curve meet the directrix in points* Z, Z′ *such that* ∠ ZSZ′ = *a right angle.*

For since $SA : SP = AZ : PZ$, as in Art. 46,

therefore $\angle XSZ = \frac{1}{2}$ supplement of XSP, [Euc. VI. A.

Similarly, $\angle XSZ' = \frac{1}{2} XSP$. [Euc. VI. 3.

By addition, $\angle ZSZ' =$ a right angle.

Hence in Art. 48 if the angles XSZ, XSZ', measured on opposite sides of the axis, be complementary, the two constructions give the same point P on the curve. Hence the simpler construction, Join AZ, $A'Z'$ intersecting in P.

50. The point Z may be any point on the directrix. Suppose Z to start from an infinite distance below the axis, and to approach X. When Z is at infinity Z' must be at X; as Z approaches the axis Z' recedes from it; when Z is at X, then Z' is at infinity. After this the points Z, Z' change sides of the axis, and each of the lengths ZX, $Z'X$ passes continuously through the values which the other had before. Hence we infer that the ellipse is symmetrical with respect to its axis. And since the angle XAP, being greater than the interior angle AXZ, is obtuse, and $XA'P$ is acute, therefore the ellipse is a closed curve lying wholly between the perpendiculars to the axis through the vertices.

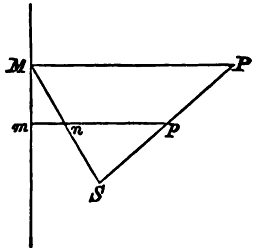
EXAMPLES.

85. Shew that $Z'Ap$, ZpA' are straight lines.

86. The straight lines joining a vertex to the ends of a focal chord intercept on the directrix a length which subtends a right angle at the focus.

87. The parallels to the axis through P, p cut ZS, $Z'S$ in points Q, q such that Qq meets Pp on the directrix, and cuts the axis at a distance equal to the semi-latus rectum from S.

¶ 51. Prop. I. *The distance of any point* P *outside a conic from the focus* S *is to its perpendicular distance from the directrix in a ratio greater than the eccentricity.*



Let SP cut the conic in p, from which point let fall a perpendicular pm on the directrix. The ratio $Sp : pm$ is equal to the eccentricity. Let pm cut SM in n. Then by parallels,

$$SP : PM = Sp : pn > Sp : pm.$$

As the figure is drawn p falls between S and P. When P lies beyond the directrix, p is to be taken in PS produced.

¶ 52. When P is within the conic it may be shewn in like manner that

$$SP : PM < Sp : pm,$$

where p is to be taken in SP produced*.

This proposition illustrates a remark made above; for it is immaterial whether the eccentricity $Sp : pm$ be less than unity or wholly unrestricted. In other words, the theorem is proved as simply for all conics at once as it could be for the ellipse separately.

Examples.

88. If an ellipse, a parabola, and a hyperbola have the same focus and directrix, the ellipse lies wholly within the parabola, and the parabola wholly within the hyperbola.

89. Conics having the same focus and directrix do not meet.

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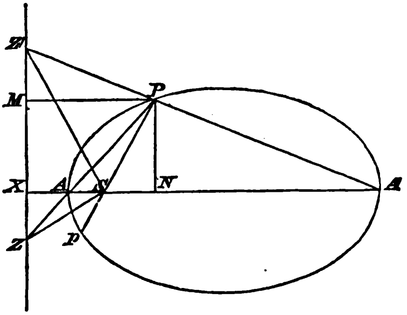
* Except when P is within the further branch of a hyperbola.

53. PROP. II. *The square of the principal ordinate* PN *of any point* P *on an ellipse whose vertices are* A, A′, *varies as* AN . NA′.



Let $A'P$, PA meet the directrix in Z', Z.

Then $PN : AN = ZX : AX$,

and $PN : A'N = Z'X : A'X$.

Compounding,

$$PN^2 : AN . A'N = ZX . Z'X : AX . A'X$$
$$= SX^2 : AX . A'X,$$

since the angle ZSZ' is a right angle. [49.

Therefore $PN^2 : AN . A'N$ is a constant ratio.

DEF. The middle point C of AA' is called the *Centre** of the ellipse. The central chord BCB' perpendicular to the axis is called the *Minor Axis*, and AA' itself is also called the *Major Axis*. CN is called the *Principal Abscissa* of P.

EXAMPLES.

90. Shew conversely that if PN^2 varies as $AN . NA'$, then $ZX . XZ'$ is constant.

91. Determine the value of this constant by making N coincide with S.

92. In Art. 44, if SD be drawn parallel to PR to meet the directrix,

$$SP : PR = L : SD.$$ [67.

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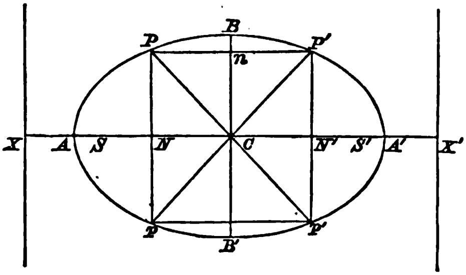
* The ellipse and the hyperbola are called *Central Conics*.

ON THE SYMMETRY OF THE ELLIPSE.

This result leads to a simple method of determining the form of the ellipse.

54. Let PN coincide with BC. Then the constant ratio $PN^2 : AN.NA'$ assumes the form $CB^2 : CA^2$.



Hence always

$$PN^2 : AN.NA' = CB^2 : CA^2,$$

or

$$PN^2 : CA^2 - CN^2 = CB^2 : CA^2.$$

When the length CN is known the length PN is known. And since PN may be measured either upwards or downwards from the axis, therefore, as we have already seen, the ellipse is divided symmetrically by its axis.

When the length PN is known the length CN is known. And since CN may be measured either to the right or to the left from C the ellipse is divided symmetrically by its minor axis.

55. From any point P let fall a perpendicular Pn on BB', and produce it to P' so that $P'n$ may be equal to Pn. If now, with some writers, we call P' the *Reflexion* of P with respect to BB', then we may say that BB' divides the figure into two parts such that one is the reflexion of the other. The ellipse must therefore have a second focus S' the reflexion of S, and a second directrix the reflexion of the first. The second focus and directrix have the same properties as

the first, and the curve may be described equally well from either of the relations

$$\begin{aligned} SP : NX &= SA : AX, \\ S'P : NX' &= S'A' : A'X' \\ &= SA : AX. \end{aligned}$$

¶ 56. It is hence evident that equal focal radii, as SP, $S'P'$, Sp, $S'p'$ and equal focal chords are such as are equally inclined to the axis.

57. Prop. III. *The sum of the focal radii to any point on the ellipse is constant.*

For $SP : NX = SA : AX = S'P : NX'$. [55.

Hence $SP + S'P : NX + NX' = SA : AX$;

or $SP + S'P : \ 2CX = SA : AX$.

Therefore $SP + S'P$ is constant.

Let P coincide with A. Then the constant value of $SP + S'P$ assumes the form $SA + S'A$, that is AA' or $2CA$.

Hence always $SP + S'P = AA' = 2CA$.

58. Take any point Q outside the ellipse. On the arc intercepted by SQ, $S'Q$ take any point P. Then $SQ + S'Q$ is greater than $SP + S'P$, or AA'. [Euc. I. 21.

For an internal point, $SQ + S'Q$ is less than AA'.

The following results should be noticed:

59. When P coincides with B, $SB + S'B = 2CA$.

Hence $SB = S'B = CA$.

60. Hence $CS^2 = SB^2 - CB^2 = CA^2 - CB^2$.

61. Also $AS.SA' = CA^2 - CS^2 = CB^2$.

62. Again, $SB : CX = SA : AX$. [Def.

Therefore $CA : CX = SA : AX$.

63. Hence $CA - SA : CX - AX = CA : CX$;

or

$$\begin{aligned} CS : CA &= CA : CX \\ &= SA : AX. \end{aligned}$$

64. Hence also $CS : CX = CS^2 : CA^2$,

and $CS . CX = CA^2$.

65. Therefore $CS . CX - CS^2 = CA^2 - CS^2$,

or $CS . SX = CB^2$.

66. The definition may be expressed in the form

$SP : PM = CS : CA$, [63.

or $SP : NX = CS : CA$.

67. If L denote the focal ordinate or semi-latus rectum, then, writing S for N in Art. 54,

$L^2 : AS . SA' = CB^2 : CA^2$,

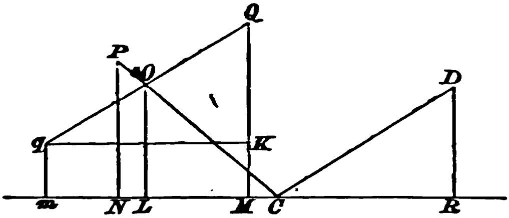
or $L^2 : \quad CB^2 \quad = CB^2 : CA^2$. [61.

Therefore CB is a mean proportional to CA and L.

Or thus: $L : SX = CS : CA$, [66.

therefore $L . CA = CS . SX = CB^2$. [65.

68. PROP. IV. *The locus of the middle points of any system of parallel chords is a straight line passing through the centre of the ellipse.*



Let Qq be any one of the system of chords parallel to the radius CD; and let QM, qm be the ordinates of Q, q.

Then $QM^2 : CA^2 - CM^2 = CB^2 : CA^2$, [54.

and $qm^2 : CA^2 - Cm^2 = CB^2 : CA^2$.

Hence $QM^2 - qm^2 : Cm^2 - CM^2 = CB^2 : CA^2$.

Bisect Qq in O. Let OL, DR be the ordinates of O, D. Draw qK parallel to the axis to meet QM.

Then[*] $$QM - qm : Cm - CM = QK : qK$$
$$= DR : CR,$$
by similar triangles QqK, DCR.

Also $QM + qm : Cm + CM = 2OL : 2CL$. [Introd.

Hence[†] $2OL \,.\, DR : 2CL \,.\, CR = CB^2 : CA^2$, from above.

Therefore, DR and CR being constant, the ratio $OL : CL$ is constant, or the locus of O is a straight line through C.

69. Hence all diameters pass through the centre. It is evident that equal diameters are such as are equally inclined to the axis. Thus in the figure on p. 30, the diameters Pp', $P'p$ are equal and equally inclined to the axis.

70. Produce CO to meet the ellipse in P. It follows at once from what is proved in Art. 68, that, PN being the ordinate of P,
$$PN \,.\, DR : CN \,.\, CR = CB^2 : CA^2.$$

The symmetry of this relation shews that if CP bisects chords parallel to CD, then CD bisects chords parallel to CP.

Def. Two diameters are *Conjugate* when each bisects chords parallel to the other. Thus the axes are conjugate.

More generally any two chords parallel to conjugate diameters may be called *Conjugate Chords*.

Supplemental Chords are such as join the ends of any diameter to some point on the curve.

71. *To shew that Supplemental Chords are Conjugate.*

In the figure of Art. 78 let SS' be a diameter of an ellipse; P any point on the curve; C the centre. Bisect SP in O and $S'P$ in O'. Then CO, which bisects both SP and SS', is parallel to PS'. Similarly CO' is parallel to SP. Hence each of the diameters CO, CO' bisects chords parallel to the other[‡], and the supplemental chords SP, $S'P$ are parallel to these diameters.

---

[*] Some of the signs may have to be changed according to the quadrants in which Q, q lie.

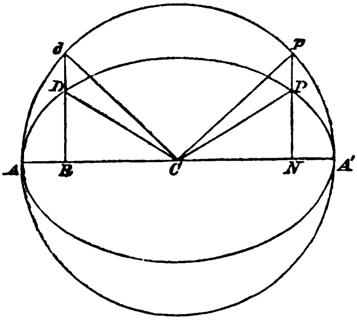
[†] Or in the usual analytical form, $\tan\theta \,.\, \tan\phi = -\frac{b^2}{a^2}$.

[‡] This proves again the result of Art. 70.

DEF. The circle described on the major axis as diameter is called the *Auxiliary Circle.*

72. PROP. V. *If the ordinate* NP *of any point* P *on the ellipse be produced to meet the auxiliary circle in* p, *then*

$$PN : pN = CB : CA.$$



For $PN^2 : AN.NA' = CB^2 : CA^2$, [54.

and $AN.NA' = pN^2$. [Euc. III. 35.

Therefore $PN^2 : pN^2 = CB^2 : CA^2$.

73. Conversely, if the ordinate NP be produced to p in the ratio $CA : CB$, the locus of p will be the auxiliary circle.

And if the ordinate pN of any point on a circle be cut in a constant ratio, the locus of the point of section will be an ellipse whose axes are in that ratio.

74. PROP. VI. *To draw conjugate diameters.*

Take conjugate radii Cp, Cd of the auxiliary circle. These include a right angle. [Euc. III. 3.

Therefore $\angle pCN =$ complement of $dCR = CdR$.

Hence the right-angled triangles pCN, CdR are similar. And, since also $Cp = Cd$, they are equivalent, so that $pN = CR$, and $dR = CN$.

Let pN, dR cut the ellipse in P, D.

Then $PN.DR : pN.dR = CB^2 : CA^2$. [72.

Hence $PN.DR : CR.CN = CB^2 : CA^2$, from above.

Therefore CP, CD are conjugate. [70.

75. Conversely, if CP, CD be conjugate, and the ordinates NP, RD meet the auxiliary circle in p, d, the angle pCd will be a right angle, and the triangles pCN, CdR will be equivalent, so that $CR=pN$, and $CN=dR$.

Hence $\quad PN:CR=CB:CA=DR:CN.$ [72.

76. PROP. VII. *The sum of the squares of conjugate diameters is constant.*

Let CP, CD be conjugate radii. Let the ordinates NP, RD meet the auxiliary circle in p, d. Then the triangles pCN, CdR are equivalent, so that

$$CR=pN, \text{ and } CN=dR. \qquad [75.$$

Therefore $CN^2+CR^2=CN^2+pN^2=CA^2$, [Euc. I. 47. since the radii CA, Cd are equal.

Also $\quad PN:CR=CB:CA=DR:CN.$ [75.

Hence $\quad PN^2+DR^2:CB^2=CN^2+CR^2:CA^2.$

But $\quad CN^2+CR^2=CA^2$, from above.

Therefore $\quad PN^2+DR^2=CB^2.$

By addition,

$$CN^2+PN^2+CR^2+DR^2=CA^2+CB^2;$$

or

$$CP^2 \quad + \quad CD^2 \quad = CA^2+CB^2.$$

77. PROP. VIII. *The parallelogram which has its sides equal and parallel to conjugate diameters is of constant area.*

For $\quad PN:pN=CB:CA=DR:dR$, as in Art. 74.

Hence the rectilinear areas

$$PCN,\ DCR,\ PDRN,$$
$$pCN,\ dCR,\ pdRN,$$

are each to each in the ratio of CB to CA.

But $\quad \triangle PCD=PDRN-PCN-DCR,$

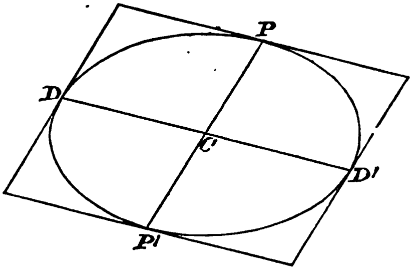
and $\quad \triangle pCd=pdRN-pCN-dCR.$

Hence $\triangle PCD$ is to pCd, that is $\frac{1}{2}CA^2$, as CB to CA.

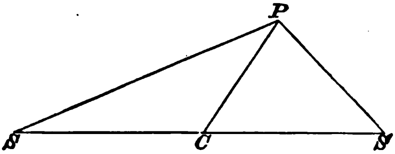
Therefore $\triangle PCD = \frac{1}{2} CA \,.\, CB.$

Describe a parallelogram by drawing parallels to conjugate diameters PCP', DCD' through their extremities. Its area is four times that of the parallelogram PD; and

$$\text{the parallelogram } PD = 2\triangle PCD = CA \,.\, CB.$$



78. Prop. IX. *If* CP, CD *be conjugate radii,*
$SP \,.\, S'P = CD^2.$



Since $(SP + S'P)^2 = (2CA)^2,$ [57.

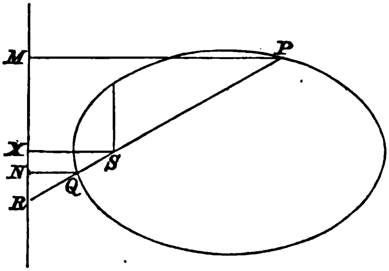
therefore $2SP \,.\, S'P + SP^2 + S'P^2 = 4CA^2.$ [Euc. II. 4.

And because C bisects SS',

therefore $SP^2 + S'P^2 = 2CS^2 + 2CP^2.$ [Introd.

Hence
$$\begin{aligned} SP \,.\, S'P &= 2CA^2 - CS^2 - CP^2 \\ &= CA^2 + CB^2 - CP^2 \quad [60. \\ &= CD^2. \quad [76. \end{aligned}$$

¶ 79. PROP. X. *The focal ordinate is a harmonic mean between the segments of any focal chord.*



Let a focal chord PQ meet the directrix in R.

Then $$SP : PR = SQ : QR, \quad [45.$$

or $$PR : QR = PR - SR : SR - QR.$$

That is to say, PR, SR, QR are in harmonical progression, the first being to the third as the difference between the first and second to the difference between the second and third.

But by parallels, if L denote the semi-latus rectum,

$$PR : SR : QR = PM : SX ; QN$$
$$= SP : L : SQ. \quad [\text{Def.}$$

Therefore SP, L, SQ are in harmonical progression.

¶ 80. The result may be written in either of the forms

$$\frac{1}{SP} + \frac{1}{SQ} = \frac{2}{L},$$
$$SP : SQ = SP - L : L - SQ,$$
$$2SP \cdot SQ = L(SP + SQ) = L \cdot PQ.$$

¶ 81. COR. If PQ, pq be any two focal chords,

$$SP \cdot SQ : Sp \cdot Sq = PQ : pq.$$

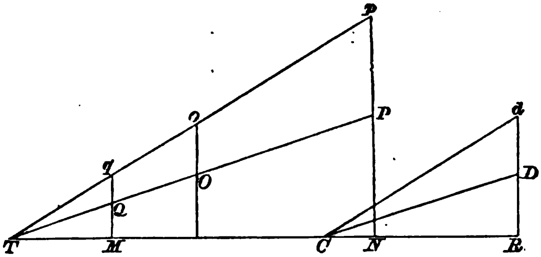
EXAMPLE.

93. Given the focus of a conic and a focal chord, find the locus of the ends of the latus rectum.

ON AUXILIARY POINTS.

DEF. If any principal ordinate NP be produced to p in the ratio $CA : CB$, then p may be called the *Auxiliary Point* of P. When P lies on the ellipse, p lies on the auxiliary circle. [73.

82. *If any number of points* O, P, Q ... *lie on a straight line which meets the axis in* T, *their auxiliary points* o, p, q ... *will lie on a straight line which meets the axis in the same point* T.



Draw the ordinates pPN, qQM. Then from the definition of auxiliary points, and by parallels,

$$pN : qM = PN : QM = NT : MT.$$

Therefore Tpq is a straight line.

And similarly it may be shewn that if O be any point on PQ, then o lies on pq.

83. *If* PQ *be parallel to* CD, *then shall* pq *be parallel to* Cd, *where* d *is the auxiliary point of* D.

Draw the ordinate dDR.

Then $pN : dR = PN : DR = NT : CR.$

Therefore pTN, dCR are similar triangles, so that Tp, Cd are parallel.

84. PROP. XI. *If a chord* PQ *passes through a fixed point* O, *the rectangle* OP . OQ *varies as the square on the parallel radius* CD.

Take auxiliary points o, p, q, d, and let the straight lines opq, OPQ meet the axis in T. [82.

Now in the triangle oTO, because pP is parallel to the side oO, therefore

$$OP : op = TP : Tp.$$

Similarly,

$$OQ : oq = TP : Tp.$$

Compounding,

$$\begin{aligned} OP . OQ : op . oq &= TP^2 : Tp^2 \\ &= CD^2 : Cd^2, \end{aligned}$$

by similar triangles TpP, CdD, their sides being parallel each to each. [83.

Now since O is fixed its auxiliary point o is fixed. Therefore $op . oq$ is constant, whether o be without or within the auxiliary circle. [Euc. III. 35, 36.

Hence, the radius Cp being constant, the ratio of $OP . OQ$ to CD^2 is constant whether O be without or within the ellipse.

85. A focal chord pSq varies as $Sp . Sq$ or CD^2. [81.

86. If POQ, $P'OQ'$ be any two chords* parallel to the radii CD, CD' and to the focal chords pq, $p'q'$, then

$$\begin{aligned} OP . OQ : OP' . OQ' &= CD^2 : CD'^2 \\ &= pq : p'q'. \end{aligned} \qquad [85.$$

¶87. COR. Hence, and from Art. 56, the chords of intersection of a conic with a circle are equally inclined to the axis. For if P, P', Q, Q' be concyclic these rectangles are equal. Hence pq, $p'q'$ are equal, and therefore equally inclined to the axis.

---

[*] As a particular case either of these chords may *coincide* with the parallel diameter.

¶88. The following enunciation suits all conics:

The rectangles contained by the segments of any two intersecting chords are to one another as the lengths of the parallel focal chords.

We shall now give a more general definition of the terms Ordinate and Abscissa.

Def. Any two straight lines being taken as *Coordinate Axes,* from any point Q draw QV parallel to one of them and terminated by the other. Then QV is called the *Ordinate* and CV the *Abscissa* of Q; and QV, CV together are called the *Coordinates of* Q.

The axes here spoken of may or may not coincide with the axes of the ellipse.

89. Prop. XII. *If* QV, CV *be the coordinates of any point* Q *on the ellipse referred to conjugate diameters* DD′, PP′, *then*

$$QV^2 : PV \,.\, VP' = CD^2 : CP^2.$$

Complete the chord QVQ' as in the figure on p. 44.

Then $QV.VQ' : PV.VP' = CD^2 : CP^2$, as in Art. 86.

But CP bisects chords parallel to CD.

Therefore $QV^2 : PV.VP' = CD^2 : CP^2$.

We may also write

$$QV^2 : CP^2 - CV^2 = CD^2 : CP^2.$$

90. If the axes of the ellipse be taken as coordinate axes, then

(i) If the abscissæ be measured on the major axis we come back to Prop. II.

(ii) If the abscissæ be measured on the minor axis, then

$$QV^2 : PV.VP' = CA^2 : CB^2.$$

91. Hence it may be shewn by the method of Art. 72 that any ordinate QV perpendicular to the minor axis is cut in the ratio $CB : CA$ by the circle on that axis. The circle on BB' is in fact a Minor Auxiliary Circle, having auxiliary properties corresponding to those of the circle on AA'.

92. PROP. XIII. *To find the length of any focal chord.*

Let PQ be a focal chord parallel to the radius CD. Denote by $2L$ the latus rectum, which is the focal chord parallel to CB.

Then $PQ : 2L = CD^2 : CB^2$ [85.

$= CD^2 : L . CA.$ [67.

Hence $PQ . CA = 2CD^2$.

¶93. PROP. XIV. *The straight lines which join adjacent extremities of any two focal chords meet two and two on the directrix.*

This will be proved in the course of Art. 105.

¶94. *The locus of the middle points of any system of parallel chords of a conic is a straight line. And the bisecting line meets the directrix on the straight line through the focus perpendicular to the chords.*

This has been proved for the parabola in Art. 6; and the first part has been proved for the ellipse in Art. 68. We shall now indicate a method of proving the proposition generally.

In the figure on p. 25 let PQ be one of a system of parallel chords. Draw a perpendicular SY to PQ. Bisect PQ in O.

Then since $SQ : QR = SP : PR,$ [44.

therefore $SP^2 - SQ^2 : PR^2 - QR^2 = SP^2 : PR^2$.

But SP^2 and SQ^2 are equal respectively to

$$PY^2 + SY^2 \text{ and } QY^2 + SY^2.$$

By subtraction

$$SP^2 - SQ^2 = PY^2 - QY^2 = PQ . 2OY. \quad \text{[Introd.}$$

Also, $PR^2 - QR^2 \quad = PQ . 2OR.$

Hence $OY : OR = SP^2 : PR^2$, from above.

Therefore $OY : OR$ is constant for parallel chords, [44.

and the locus of O is a straight line meeting the directrix in the same point with SY.

95. Let two diameters meet the directrix in p, d. Then if Cd bisects chords parallel to Cp, it may be easily shewn that S is the Orthocentre of the triangle pCd, and hence that Cp bisects chords parallel to Cd.

Also that $pX \,.\, dX : CX^2 = CB^2 : CA^2$, [64, 5.
which agrees with Art. 70.

¶ 96. *To shew that a conic is concave to its axis.*

In the figure on p. 25, since SR bisects the angle supplementary to PSQ, therefore if O be any point in the chord PQ,

$$SO : OR < SP : PR. \qquad \text{[Introd.}$$

But if O' be the point, above the axis, in which the ordinate of O cuts the conic, then

$$SO' : OR = SP : PR. \qquad \text{[44.}$$

Hence SO', being greater than SO, lies above it; or every arc PQ, however small, is more remote from the axis than the chord of the arc.

CHAPTER IV.

TANGENT-PROPERTIES OF THE ELLIPSE.

97. THE tangents at the ends of any diameter are parallel to the ordinates of that diameter and to one another. Conversely, parallel tangents touch the curve at opposite ends of a diameter. [17.

98. The tangents at the ends of either axis are parallel to the other.

99. The sides of a parallelogram described as in Art. 77 are tangents. Hence

The area of a circumscribing parallelogram which has its sides parallel to conjugate diameters is constant, and equal to the rectangle contained by the axes.

100. Let the normal at P meet DD' in F.

$$\text{Then}\qquad PF \,.\, CD = \text{parallelogram } PD$$
$$= CA : CB.$$

101. In Art. 86 it is shewn that the rectangles contained by the segments of any two intersecting chords are to one another as the rectangles contained by the segments of any other two chords parallel to the former. Hence if two of the chords, moving parallel to themselves, become tangents TP, Tp, and the other two chords become diameters DD', dd' parallel to those tangents, then the proposition assumes the form

$$TP^2 : Tp^2 = CD^2 : Cd^2.$$

Hence any two tangents TP, Tp are as the parallel radii CD, Cd.

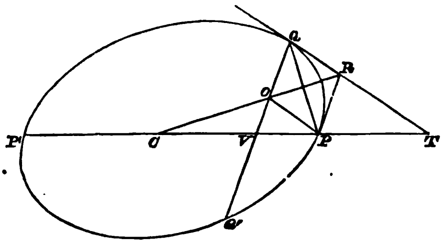
102. In Art. 82 suppose the points Q, q to become coincident with P, p. Then the lines PQ, pq become tangents to the ellipse at P and to the auxiliary circle at p respectively, and they always meet in a point T on the axis. Hence, the radius Cp being perpendicular to the tangent pT,

$$CN \, . \, CT = Cp^2 = CA^2.$$

We proceed to prove a more general theorem which includes this.

103. PROP. XV. *If* CV *be the abscissa of a point* Q, *the tangent at which meets the radius of abscissæ* CP *in* T, *then*

$$CV \, . \, CT = CP^2.$$



The tangent at P is parallel to QV. Let it meet QT in R. Complete the parallelogram $QRPO$, viz. by drawing PO parallel to RQ. Then the diagonal RO bisects PQ.

But PQ is a chord of contact, viz. of tangents RP, RQ; and therefore its bisector RO is a diameter. [22.

Let it be produced to the centre C.

Then by parallels $CV : CP = CO : CR$
$= CP : CT$.

Therefore $CV \, . \, CT = CP^2$.

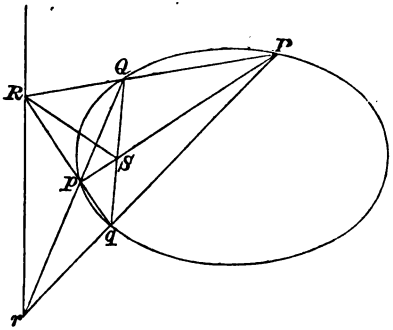
104. If P coincide with A, and N be written for V, then

$$CN \, . \, CT = CA^2.$$

The corresponding property of the minor axis may be expressed,

$$Cn \, . \, Ct = CB^2.$$

¶105. PROP. XVI. *Tangents at the ends of a focal chord meet on the directrix; and every tangent, measured from the curve to the directrix, subtends a right angle at the focus.*



(i) Let *Pp*, *Qq* be any two focal chords; and let *PQ* meet the directrix in *R*. Then from the definition,

$$SP : SQ = PR : QR; \qquad [45.$$

or *R* lies on the bisector of ∠ *pSQ*. [Euc. VI. A.

So too it may be shewn that *pq* meets the directrix on the bisector of ∠ *pSQ*. That is to say, *PQ*, *pq* meet the directrix in the same point *R*.

(ii) Now let *Pp*, *Qq* be adjacent chords. And let *Qq* turn about *S* until it coincides with *Pp*. Thus the joining lines *PQ*, *pq*, which always meet on the directrix, become the tangents at *P*, *p*. And since *SR* is always equally inclined to *Sp*, *SQ*, therefore in the limit, when *SQ* coincides with *SP*, it makes equal angles with *Sp*, *SP*. That is to say, *SR* is at right angles to *Pp*, the chord of contact of tangents from *R*. Compare the next figure.

¶106. Conversely, the chord of contact of the tangents drawn from any point on the directrix passes through the focus.

¶107. DEF. The point of intersection of the tangents at the ends of any chord is said to be the *Pole* of the chord; and the chord of contact of the tangents from any point is said to be the *Polar* of the point.

¶108. Prop. XVII. *If from any point* T *on the tangent at* P *to a conic, perpendiculars* TL, TN *be let fall on* SP *and the directrix, then*

$$SL : TN = SA : AX.$$

As in Art. 28, $SL : SP = RT : RP$

$= TN : PM.$

Alternately, $SL : TN = SP : PM$

$= SA : AX.$

¶109. Cor. Conversely, to draw tangents to a conic from a given point *T*. With radius *SL* determined from the proposition describe a circle about *S*. Draw tangents *TL*, *TM* to the circle. Then *SL*, *SM* meet the conic in the required points of contact.

¶110. Prop. XVIII. *The focal radius to the pole* T *of any chord of a conic makes equal angles with the radii to its ends* P, Q*.

With the construction of Art. 29,

$$SL : TN = SA : AX,$$

since *T* lies on the tangent at *P*.

And $SM : TN = SA : AX,$

since *T* lies on the tangent at *Q*.

Hence, in the right-angled triangles *SLT*, *SMT*, the sides *SL*, *SM* are equal. And *ST* is common. Therefore the angles *TSL*, *TSM* are equal, or *ST* makes equal angles with *SP*, *SQ*.

¶111. Def. The point in which a chord meets the directrix is called the *Foot* of the chord†.

¶112. *The focal radii to the foot* R *and the pole* T *of any chord* PQ *include a right angle.*

For *SR*, *ST* bisect supplementary angles *pSQ*, *PSQ*.

[105, 110.

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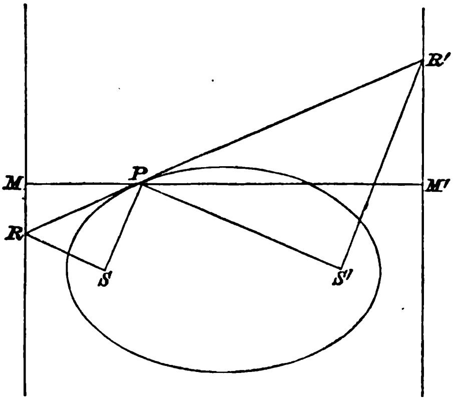
* The enunciation of Art. 29 applies in all cases except when the tangents are drawn to *opposite branches* of a hyperbola. It will appear in the sequel that tangents so drawn subtend supplementary angles at either focus.

† This definition is borrowed from Mr H. G. Day's treatise on the Ellipse.

113. PROP. XIX. *In the ellipse the tangent at* P *is equally inclined to* SP, S'P; *and the normal at* P *bisects the angle* SPS'.



(i) Let a perpendicular through P meet the directrices in M, M'; and let R, R' be the feet of the tangent. [111.

Then $SP : S'P = PM : PM'$ [Def.

$= PR : PR'$;

by similar triangles $PMR, PM'R'$.

Also $\angle PSR =$ a right angle $= PS'R'$. [105.

Therefore in the triangles $SPR, S'PR'$, the angles at P are equal. That is to say, the tangent at P makes equal angles with $SP, S'P$; or, as in fig. Art. 115, bisects the angle between SP produced and $S'P$.

(ii) In the next figure let tPT be the tangent and PG the normal.

Then $\angle tPG =$ a right angle $= TPG$,

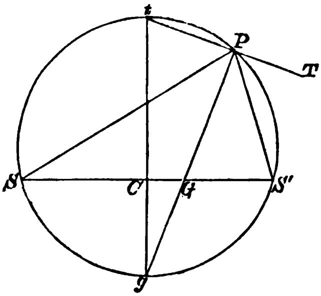
or $\angle SPt + SPG = S'PG + S'PT$.

Therefore, subtracting the equal angles $SPt, S'PT$,

$\angle SPG = S'PG$,

or the normal bisects the angle SPS'.

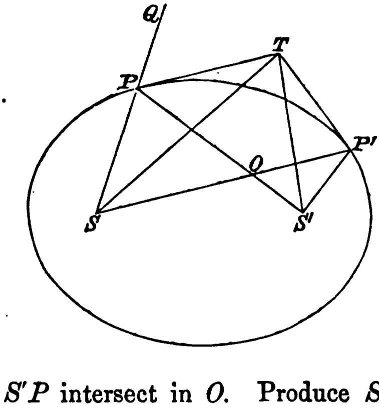
114. PROP. XX. *The circle through the foci and any point* P *on the ellipse meets the minor axis on the tangent and the normal at* P.



Let the circle through SPS' cut the minor axis in g, t. Then the equal arcs gS, gS' subtend equal angles at P. Therefore Pg is the normal at P. [113.

Also gt, which bisects SS' at right angles, is a diameter of the circle, and therefore subtends a right angle at P. Hence Pt, being at right angles to the normal, is the tangent at P.

115. PROP. XXI. *If* T *be the pole of the chord* PP′, *then* ∠ STP = S′TP′.



Let SP', $S'P$ intersect in O. Produce SP to Q. Then TP, TS bisect the angles $S'PQ$, PSP'. [113, 110.

Hence $\angle\, TPQ - TSQ = \frac{1}{2}\, OPQ - \frac{1}{2}\, OSP$;

or $\angle\, STP = \frac{1}{2}\, POS.$ [Euc. I. 32.

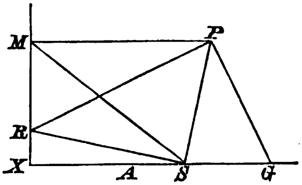
Similarly $\angle\, S'TP' = \frac{1}{2}\, P'OS'.$

Hence, the vertical angles at O being equal,

$$\angle\, STP = S'TP'.$$

¶116. Prop. XXII. *If the normal to a conic at* P *meet the axis in* G, *then*

SG : SP = SA : AX.



Let the tangent at P meet the directrix in R. Then the circle on PR as diameter passes through M the foot of the perpendicular from P to the directrix. It also passes through S, since PSR is a right angle. [105.

Now PG is at right angles to PR and touches the circle. [Euc. III. 16. Cor.

Therefore $\angle\, SPG = SMP$ in the alternate segment.

And $\angle\, PSG = SPM$, by parallels.

Hence the triangles SPG, SMP are similar, so that

$$SG : SP = SP : PM$$
$$= SA : AX.$$

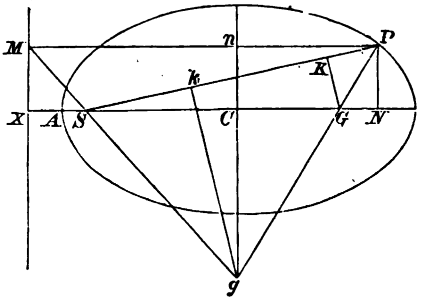
¶117. Conversely, when P is given this relation determines G, and the normal at P may be drawn.

118. In the ellipse,

$$SG : SP = SP : PM = CS : CA.$$

Also $$SG : PM = CS^2 : CA^2 = CS : CX.$$ [64.

Conversely, if SG, drawn as in the figure, satisfies this relation, then PG is the normal at P. Hence the following construction for the normal at P:



Let fall a perpendicular PM *on the directrix, and let* MS *meet the minor axis in* g. *Then* Pg *is the normal.*

For let it cut the axis in some point G; and let the minor axis cut PM in n. Then the parallels SCG, MnP are cut in the same ratio, so that

$$\begin{aligned} SG : PM &= CS : nM \\ &= CS : CX. \end{aligned}$$

119. Also, by parallels and from what precedes,

$$\begin{aligned} Pg : Gg &= PM : SG \\ &= CX : CS. \end{aligned}$$

Therefore $$\begin{aligned} Pg : PG &= CX : SX \\ &= CA^2 : CB^2. \end{aligned}$$ [64, 5.

120. Again, by similar triangles PGN, gPn, and since $Pn = CN$,

$$NG : CN = PG : Pg = CB^2 : CA^2,$$

or the subnormal varies as the abscissa.

¶121. Prop. XXIII. *If from the points* G, g *in which the normal at* P *meets the axes, perpendiculars* GK, gk *be let fall on* SP, *then will* PK* *and* Pk *be constant.*

(i) Draw the principal ordinate PN. Then by similar right-angled triangles SKG, SNP,

$$SK : SN = SG : SP$$
$$= SP : NX. \qquad [116.$$

Hence $SP - SK : NX - SN = SP : NX$,

or PK is to SX as the eccentricity.

Therefore $PK = \frac{1}{2}$ latus rectum; [Def.

or $PK . CA = CB^2$. [67.

(ii) Also $PK : Pk = PG : Pg$, by parallels,

therefore $PK . CA : Pk . CA = CB^2 : CA^2$. [119.

And the antecedents being equal, the consequents are equal.

Therefore $Pk = CA$.

Examples.

94. Shew that NK is parallel to SM, and deduce that

$$PK : SX = SP : PM.$$

95. If gk, gk' be perpendiculars to SP, SP', then Pk, Sk are equal respectively to Pk', $S'k'$. Deduce that

$$Pk = \frac{1}{2}(SP + PS') = CA.$$

96. If a circle has double contact with a conic, a chord of the circle through the focus and either point of contact has one of two constant values.

97. If it passes through one focus determine the point of contact; and if through both foci, the eccentricity.

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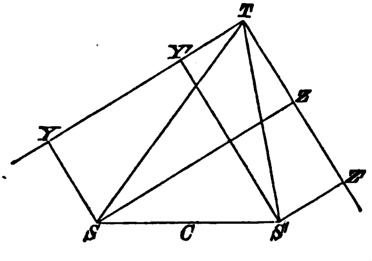
* The first part only of this proposition applies to all conics. The second part applies to central conics.

122. **Prop. XXIV.** *The perpendiculars from the two foci of an ellipse upon any tangent contain a constant rectangle.*



Let fall perpendiculars SY, $S'Y'$ and SZ, $S'Z'$ upon two tangents which meet in a point T.

Then $\angle STY = S'TZ'$. [115.

Hence by similar triangles STY, $S'TZ'$, and by similar triangles STZ, $S'TY'$,

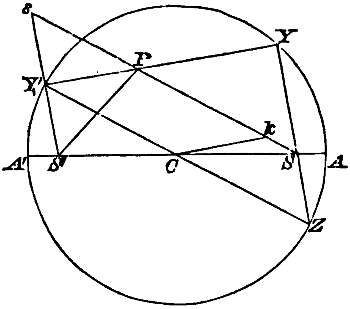
$$\begin{aligned} SY : S'Z' &= ST : S'T \\ &= SZ : S'Y'. \end{aligned}$$

Let TZ become parallel to the axis; then SZ, SZ' become each equal to CB.

Hence always $SY : CB = CB : S'Y'$,

or $SY \,.\, S'Y' = CB^2$.

123. **Prop. XXV.** *The feet of the focal perpendiculars on any tangent to an ellipse lie on the auxiliary circle.*



Let SY, $S'Y'$ be perpendiculars on the tangent at P.

Let SP, $S'Y'$ meet in s. Then the tangent PY' bisects the angle $S'Ps$. [113.

Hence the triangles sPY', $S'PY'$ have their angles at P equal, and also their angles at Y. And the side PY' is common.

Therefore $sP = S'P$, and $sY' = S'Y'$.

Hence CY' bisects both sS' and SS'.

Therefore CY' is parallel to Ss and equal to $\frac{1}{2}Ss$.

Now $$Ss = SP + Ps = SP + PS', \text{ from above,}$$ $$= 2CA.$$ [57.

Therefore $CY' = CA$; or Y' lies on the auxiliary circle.

So too does Y.

124. Conversely, the straight line drawn from any point Y on the auxiliary circle at right angles to SY touches the ellipse.

125. Hence every focal chord meets the auxiliary circle in points Y, Z which lie on parallel tangents; and the rectangle contained by the focal perpendiculars SY, SZ on parallel tangents is equal to $AS \,.\, SA'$ or CB^2. [61.

126. Hence another proof of Prop. 24, since evidently $SZ = S'Y'$.

127. COR. Since CY', SP are parallel, therefore, if the diameter parallel to PY meet SP in k,

$$Pk = CY' = CA.$$

128. Hence k is the foot of the perpendicular let fall on SP from the point in which the normal at P meets the minor axis. [121.

129. PROP. XXVI. *Tangents at right angles meet on a fixed circle.*

Draw focal perpendiculars SY, $S'Y'$ and SZ, $S'Z'$ on two tangents which intersect at right angles, viz. in T. The feet of these perpendiculars lie on the auxiliary circle. [123.

Now $CT^2 = CA^2 + TZ \,.\, TZ'$, as in Euc. III. 36.

And the opposite sides of the rectangles ST, $S'T$ being equal,

$$TZ \,.\, TZ' = SY \,.\, S'Y' = CB^2. \qquad [122.$$

Therefore $\quad CT^2 = CA^2 + CB^2,$

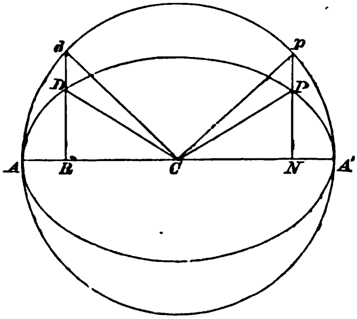
or the locus of T is a circle.

130. Def. The circle which is the locus of the intersection of tangents at right angles is called the *Director Circle.*

131. Prop. XXVII. *If the normal at* P *meet the axes in* G, g, *and* CD *be the radius conjugate to* CP, *then*

$$PG : CD = CB : CA,$$

and $$Pg : CD = CA : CB.$$



Use the figure of Art. 72, and supply the line PGg. Draw Pn perpendicular to the minor axis.

(i) Then, since CD is parallel to the tangent at P, it is perpendicular to the normal.

Therefore $\angle\, PGN =$ complement of $DCR = CDR$.

Hence, by similar right-angled triangles PGN, CDR,

$$\begin{aligned} PG : CD &= PN : CR \\ &= CB : CA. \qquad [75. \end{aligned}$$

(ii) Again, the triangle gPn is similar to PGN or CDR.

Hence $$\begin{aligned} Pg : CD &= Pn : DR \\ &= CN : DR, \text{ by parallels,} \\ &= CA : CB. \qquad [75. \end{aligned}$$

132. Hence $\quad PG \,.\, Pg = CD^2.$

133. Also, as has been proved,

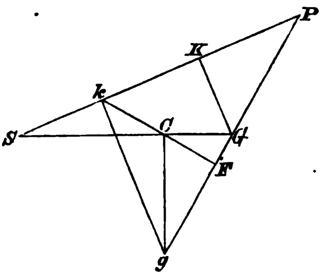
$$PG : Pg = CB^2 : CA^2. \qquad [119.$$

134. PROP. XXVIII. *If the normal at* P *cut the axes in in* G, g, *and the diameter conjugate to* CP *in* F, *then*

$$\mathrm{PF} \,.\, \mathrm{Pg} = \mathrm{CA}^2,$$

and

$$\mathrm{PF} \,.\, \mathrm{PG} = \mathrm{CB}^2.$$



(i) Let the diameter conjugate to CP meet SP in k. Then gk is perpendicular to SP. [128.

Also, since the diameter CF is parallel to the tangent at P, it meets the normal at right angles, viz. in F.

Hence $PF \,.\, Pg = Pk^2 = CA^2.$ [121.

(ii) And, since the angles at K, F are right angles, the points G, K, k, F are concyclic.

Therefore $PF \,.\, PG = PK \,.\, Pk = CB^2.$ [121.

135. Or thus:

Let the tangent at P meet the axes in T, t. Let fall perpendiculars PN, Pn on the axes, and produce them to meet the diameter CF in M, m. Then the angles at N, F, n being right angles, F, G, N, M are concyclic; and F, g, m, n are concyclic.

Hence $PF \,.\, PG = PN \,.\, PM = Cn \,.\, Ct$, by parallels,

$= CB^2.$ [104.

And $PF \,.\, Pg = Pn \,.\, Pm = CN \,.\, CT$, by parallels,

$= CA^2.$ [104.

EXAMPLES.

98. The sides AD, DC of a rectangle $ABCD$ are divided into the same number of equal parts, and straight lines are drawn from B, A respectively to the points of section. Shew that corresponding lines in the two series meet on an ellipse whose axes are equal to the sides of the rectangle.

99. Shew how to draw equal conjugate diameters. [74.

100. Apply Art. 94 to prove that a central conic is symmetrical with respect to its minor axis.

101. If SBS' be a right angle what is the eccentricity?

102. When is the angle SPS' a maximum?

103. The segments of a focal chord subtend equal angles at X.

104. If chords PR, QR be produced to meet the directrix in p, q, the angle between the focal radii to p, q will be equal to half the angle between the focal radii to P, Q.

105. Two ellipses whose major axes are equal have a common focus; prove that they intersect in two points only.

106. The major axis is the maximum chord of an ellipse, and the minor axis its least diameter.

107. Given one focus of an ellipse, a point on the curve, and the length of the axis; shew that the loci of the other focus and of the centre are circles.

108. The circle inscribed in the triangle SPS' touches SP in M, and SS' in N. Prove that $PM = A'S$, and $AM = SP$.

109. What is the locus of the centre of a circle which touches two fixed circles?

110. Two circles have their centres fixed and the sum of their radii constant. Find the locus of the centre of a circle of given radius which touches them both.

111. The common chord of two ellipses having the same focus passes through the intersection of the corresponding directrices.

112. Given an ellipse, shew how to find C and S.

113. Given S' and a chord through S, find S.

114. Two conjugate diameters being given in position and magnitude, determine the axes.

115. The locus of a point which cuts parallel chords of a circle in a given ratio is an ellipse having double contact with the circle.

116. The circle on a focal radius touches the auxiliary circle.

117. The common tangents of the auxiliary circle and the circles on SP, $S'P$ intersect on the ordinate of P.

118. Conjugate diameters meet the directrix at distances from the axis which contain a constant rectangle.

119. A focal chord and its diameter meet the directrix at distances from the axis which contain a constant rectangle.

120. The centre of a focal chord traces a similar ellipse.

121. The centre of a chord which cuts the axis in a fixed point describes an ellipse. [53.

122. A straight line equal to the radius of a circle slides with one end on a fixed diameter and the other end P on the convex side of the circumference. Shew that the coordinates of a point Q in the line vary as those of P, and hence that Q traces an ellipse.

123. Shew that $SP - CA$ varies as the abscissa of P, and that

$$(SP - CA)^2 + (CA - SD)^2 = CS^2. \qquad [76.$$

124. The parallelogram described on conjugate diameters as diagonals is of constant area.

125. A diameter of an ellipse varies inversely as the perpendicular focal chord of the auxiliary circle.

126. If the radii CP, CQ be at right angles,

$$\frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{1}{CA^2} + \frac{1}{CB^2}.$$

127. The sum of conjugate diameters is a maximum when they are equal.

128. When is the sum of conjugate diameters a minimum?

129. SQ, $S'Q$ being perpendiculars to conjugate diameters, the locus of Q is a concentric ellipse.

130. The perpendiculars to the axes from the points in which a common diameter meets the two auxiliary circles intersect two and two on the ellipse. [91.

131. The common diameters of equivalent and concentric ellipses are at right angles. [87.

132. If PP', DD' be conjugate diameters, then PD, PD' are proportional to the diameters parallel to them.

133. The sum of conjugate focal chords is constant.

134. A point in a straight line which slides between two fixed straight lines at right angles traces an ellipse.

135. Parallel diameters of similar and similarly situated conics bisect the same systems of parallel chords.

136. With the pole of a chord as centre a circle can be described touching the four focal radii to the ends of the chord.

137. A circle can be drawn through the foci and the intersections of any tangent with the tangents at the vertices.

138. The equi-conjugate diameters coincide in direction with the diagonals of a rectangle formed by the tangents at the ends of the axes.

139. The central perpendicular on the tangent varies inversely as the conjugate diameter.

140. If the tangent and ordinate at Q meet PP' the diameter of abscissæ in T, V, shew by Art. 101 that

$$TP : TP' = PV : VP'.$$

141. Shew directly by the method of Art. 103 that

$$TC \,.\, TV = TP \,.\, TP'.$$

142. If a chord of a conic subtends a constant angle at the focus, its envelope and the locus of its pole are conics having the same focus and directrix. [108.

143. The vertex of a circumscribed triangle whose base subtends a constant angle at the focus is a conic.

144. Shew by equating the angles of the quadrilaterals $SPTP'$, $S'PTP'$ to eight right angles that the external angle between the tangents is equal to half the sum of the angles which their chord of contact subtends at the foci. [115.

145. What is the corresponding theorem when the direction of PP' falls between the foci?

146. Any tangent meets parallel tangents on conjugate diameters.

147. If the tangent at P meets parallel tangents in Q, R, the rectangle $PQ \,.\, PR$ is equal to CD^2.

148. The polar of a point on the directrix passes through S.

149. The radii from a focus to the ends of a diameter make equal angles with the tangents at those points.

150. The straight lines joining the feet and the poles of any two chords subtend equal angles at the focus. [111.

151. The intercept on a tangent by tangents at the ends of a focal chord subtends a right angle at the focus.

152. If s, s' be the reflexions of S, S' with respect to a tangent through P the triangles SPs', sPS' will be equivalent.

153. A parallel to SP from S' meets SY on a circle.

154. Shew that $SY \,.\, CD = SP \,.\, CB$.

155. Also that $SY^2 : CB^2 = SP : 2CA - SP$.

156. The ordinate bisects the angle YNY', and the points Y, N, C, Y' are concyclic.

157. Shew that PG is bisected by SY', and by $S'Y$; and that it is a harmonic mean to SY, $S'Y'$.

158. Tangents being drawn from any point on a circle through the foci, shew that the bisectors of the angles between them pass through fixed points.

159. If the tangent and normal meet either axis in T, G, then

$$CG \,.\, CT = CS^2.$$

160. The bisectors of the angles between the tangents from any point are tangent and normal to the confocals through that point.

161. In Art. 114 shew that $PG.Pg$ is equal to $SP.PS'$, and deduce the theorems of Art. 131. [78, 119.

162. If CP be conjugate to the normal at Q, CQ will be conjugate to the normal at P.

163. If the tangent at P and its diameter meet the axis and directrix in T, D, then DT is parallel to PS.

164. Shew how to draw tangents to an ellipse from a point on the auxiliary circle, or from any other external point.

165. The pole of the tangent at P with respect to the auxiliary circle lies on the ordinate of P.

166. If TP, TP' be the tangents in the first case, SP will be parallel to $P'T$.

167. A circumscribing parallelogram which has two corners on the directrices has the other two on the auxiliary circle.

168. If an ellipse inscribed in a triangle has one focus at the orthocentre, the other focus will be at the centre of the circumscribed circle.

169. If an ellipse slides between two straight lines at right angles the locus of its centre is a circle.

170. The straight line joining the foci subtends at the pole of a chord half the sum or difference of the angles which it subtends at the extremities of the chord.

171. The portion of a normal chord intercepted between the directrices subtends at the pole of the chord half the sum of the angles which the straight line joining the foci subtends at the extremities of the chord. [112.

172. If a chord be produced to meet the directrices, the parts produced will subtend equal angles at the pole.

173. Supplemental chords equally inclined to the curve have their poles on the director circle. What is the corresponding property of the parabola?

174. The sum of two chords thus drawn is constant.

175. The normal at A is equal to L. [121.

CHAPTER V.

CHORD-PROPERTIES OF THE RECTANGULAR HYPERBOLA.

136. THE hyperbola has been defined as a conic whose eccentricity exceeds unity. When the eccentricity is equal to $\sqrt{2}$ * the hyperbola is called *Rectangular*, and also *Equilateral*, for reasons which will appear. [155.

137. In the next figure, let S be the focus, and X the point in which the axis meets the directrix. Take C in SX produced; such that $CX = SX$. On the axis measure CA, CA' mean proportionals to CS, CX. It will be seen from Prop. I. that A, A' are the *Vertices*, or points in which the curve cuts the axis. The point C, which bisects AA', is the centre.

138. DEF. The circle on AA' is called the *Auxiliary Circle.*

Its diameter BB' perpendicular to AA' is called the *Conjugate Axis*, and AA' is called the *Transverse Axis.* These are also, though equal, called the *Minor* and *Major Axis.*

139. Since $CS = 2CX = 2SX$, by construction,

and $CA^2 = CS \,.\, CX$,

therefore $CS^2 = 2CA^2$,

and $CA^2 = 2CX^2 = 2SX^2$.

140. The square of the semi-latus-rectum is from the definition equal to $2SX^2$, and therefore to CA^2. Hence the latus-rectum is equal to the axis.

141. Also $SA \,.\, SA' = CS^2 - CA^2 = CA^2$. [139.

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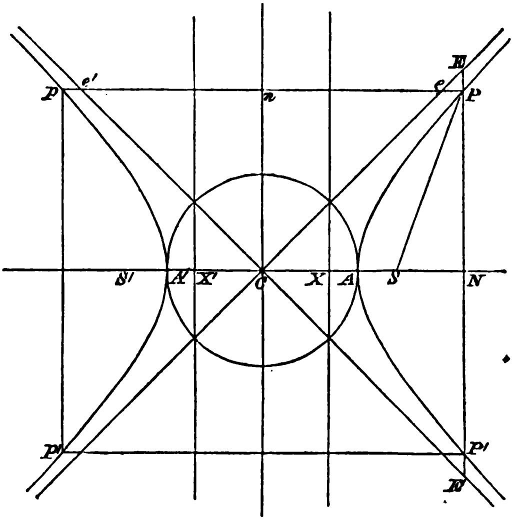
* The ratio of the diagonal to the side of a square.

142. Prop. I. *If* PN *be the principal ordinate of any point* P *on the curve, then*

$$CN^2 - PN^2 = CA^2.$$



Since $\qquad PN^2 + SN^2 = SP^2 = 2NX^2,$ [Def.

and $\qquad CN^2 + SN^2 = 2CX^2 + 2NX^2,$ [Euc. II. 10.

X being the middle point of CS,

therefore $\qquad CN^2 - PN^2 = 2CX^2 = CA^2.$ [139.

143. If Cn be the abscissa of P measured along the minor axis, then

$$Pn^2 - Cn^2 = CA^2.$$

144. Since $\quad PN^2 = CN^2 - CA^2,$
the ordinate PN is equal to the tangent NT from N to the auxiliary circle. Hence the circle described about N as centre, with radius NT, cuts a perpendicular to the axis through N in points P, P' which lie on the curve.

Thus, by taking successive positions of N, we may determine any number of pairs of points on the curve.

145. *On the form of the curve.*

It is evident from the proposition that the least value of CN is CA. Hence no part of the curve can lie between the tangents at A, A' to the auxiliary circle. Through C draw straight lines CE, CE', each inclined at half a right angle to the axis. Let EN, the ordinate of any point E on one of these lines, cut the curve in P. Then, since $EN = CN$, and from the proposition PN is less than CN, therefore PN is less than EN. Hence the curve lies wholly within the angle ECE' and the angle vertically opposite. And it spreads out to an infinite distance from both axes at once, since as PN increases from zero to infinity CN increases from CA to infinity. Like the ellipse, it is symmetrical with respect to both axes, since when the lengths PN, CN at any point P of the curve are given, we may measure NP either upwards or downwards from the axis, and CN either to the right or to the left from C. Hence the curve has a second focus S' the *Reflexion* of S, and a second directrix the *Reflexion* of the first. [55.

146. It is evident that diameters and focal chords equally inclined to the axis are equal.

147*. Prop. II. *The difference of the focal radii to any point on the curve is constant.*

With the same construction as above,

$$S'P \sim SP : NX' \sim NX = SA ; AX,$$

or

$$S'P \sim SP : \quad 2CX \quad = SA : AX.$$

Therefore the difference of $S'P$, SP is constant.

148. Let P coincide with A. Then the constant value of $S'P \sim SP$ assumes the form $S'A \sim SA$, that is AA' or $2CA$.

149. Hence

$$SP^2 + S'P^2 - 2S'P \, . \, S'P = 4CA^2 = 2CS^2. \qquad [139.$$

Therefore $2CS^2 + 2SP \, . \, S'P = S'P^2 + SP^2$

$$= 2CP^2 + 2CS^2, \qquad \text{[Introd.}$$

since C bisects SS'.

Therefore $\qquad SP \, . \, S'P = CP^2.$

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[*] This applies as it stands to the hyperbola generally.

150. Def. A line to which a curve approximates continuously and indefinitely, but which it does not meet at a finite distance, is called an *Asymptote* of the curve.

151. Prop. III. *To find the asymptotes of the rectangular hyperbola.*

Through C draw two straight lines each inclined at half a right angle to the axis; and let them meet the principal ordinate PN of any point P on the curve in E, E'. Suppose E to lie on the same side of the axis with P. [Fig. Art. 142.

Then $$CN = EN = E'N.$$

Hence $$EP \, . \, E'P = EN^2 - PN^2 = CN^2 - PN^2$$
$$= CA^2. \qquad [142.$$

Therefore as $E'P$ increases EP decreases; and it is evident that it may decrease continuously and indefinitely with the increase of CN.

Hence CE is an asymptote. So too is CE'.

152. In like manner it may be shewn that if an ordinate Pn of the minor axis meet the asymptotes in e, e', then
$$eP \, . \, e'P = CA^2. \qquad [143.$$

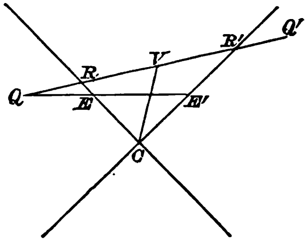
153. Cor. The intercepts EP, $E'P$, and likewise eP, $e'P$, are equal.

154. Cor. From P, which may be any point on the curve, draw perpendiculars Px, Py to CE, CE'. Then in the isosceles right-angled triangles PxE, PyE',
$$Px^2 = \frac{1}{2} PE^2, \text{ and } Py^2 = \frac{1}{2} PE'^2.$$

Hence $$Px \, . \, Py = \frac{1}{2} PE \, . \, PE' = \frac{1}{2} CA^2.$$

155. The asymptotes are at right angles. Hence the name *Rectangular* Hyperbola. The name *Equilateral* Hyperbola has reference to the equality of the axes. [138.

156. Prop. IV. *If a chord* QQ′ *always parallel to itself cuts the asymptotes in* R, R′, *the rectangle* QR . QR′ *is constant.*



Draw QEE', parallel or perpendicular to the axis according as Q, Q' do or do not lie on opposite branches, and let it meet the asymptotes in E, E'. Then the triangle EQR is always similar to itself. So too is $E'QR'$.

Therefore QR varies as QE,

and QR' varies as QE'.

Hence $QR . QR'$ varies as $QE . QE'$, or CA^2,

and is therefore constant. [152.

157. Cor. Hence $QR . QR' = Q'R . Q'R'$. Therefore the intercepts QR, $Q'R'$ are equal, QQ' being any chord.

158. Cor. The diameter bisecting QQ' bisects RR'.

Examples.

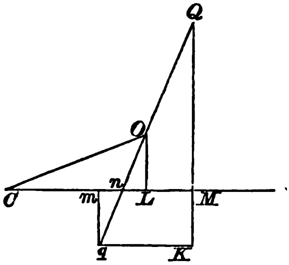
176. If AA' be a diameter of a circle and PQ one of its ordinates, then will AP, $A'Q$ intersect on a rectangular hyperbola.

177. Tangents to a parabola which include half a right angle intersect on a rectangular hyperbola.

178. If a parallel to an asymptote meet the curve in P and a principal double ordinate Qq in O, then

$$QO . Oq = 4 \triangle OCP.$$

159. **Prop. V.** *The locus of the middle points of any system of parallel chords is a straight line through the centre.*



Let Qq be any one of a system of parallel chords*; CM, Cm the abscissæ of its extremities. Let O be the centre of the chord, and OL its ordinate. Draw qK parallel to the axis to meet QM.

Then $\quad QM^2 + CA^2 = CM^2,\quad$ [142.

and $\quad qm^2 + CA^2 = Cm^2.$

Therefore $\quad QM^2 - qm^2 = CM^2 - Cm^2,$

or $\quad QM - qm : CM + Cm = CM - Cm : QM + qm.$

Therefore† $\quad 2OL : 2CL = qK : QK,\quad$ [Introd.

But the ratio $qK : QK$ is constant for parallel chords.

Therefore also $OL : CL$ is constant, and the locus of O is a straight line through C.

160. **Prop. VI.** *Conjugate diameters make complementary angles with either axis.*

For $\quad OL : CL = qK : QK$, as in Prop. v.,

$$= nM : QM,$$

if n be the point in which Qq meets the axis.

---

* Whether meeting one branch only or both branches.

† See the first note on p. 33.

Hence the right-angled triangles QnM, COL are similar, so that

$$\angle QnM = COL = \text{complement of } OCL;$$

or if CD be parallel to qQ, then CO, CD make complementary angles with CA, and therefore also with CB. [Fig. p. 68.

161. Cor. Conjugate radii CP, CD are equally inclined to the axes each to each.

162. Cor. Also they are equally inclined to either asymptote.

163. Hence it appears that the acute angle between two conjugate diameters lies wholly in one of the four quadrants bounded by the axes. Also that one of every two conjugate diameters makes with the axis an acute angle greater than half a right angle, and therefore evidently does not meet the curve.

Likewise one of every two diameters at right angles does not meet the curve.

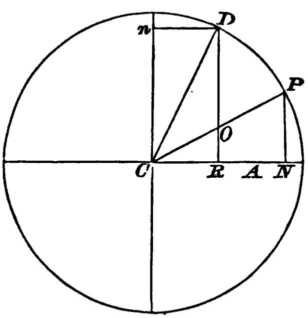
164. Every diameter which makes with the axis an acute angle less than half a right angle meets the curve. For, in the figure on p. 62 draw a diameter cutting the intercept EP in O. Then however small an angle the diameter makes with CE, the length EO may be increased indefinitely by making the ordinate recede from C.

Therefore EO becomes at some point equal to and then greater than EP, which diminishes continually. That is to say, the diameter cuts the curve.

165. Hence it appears that no straight line through C except CE and CE' can be an asymptote. In like manner it may be shewn that no other straight line can be an asymptote. Also that a straight line parallel to an asymptote cuts the curve once and does not meet it again.

166. **Def.** It is convenient to define the *length* of a diameter which does not meet the curve as equal to the length of its conjugate*; but we shall avoid speaking of the "extremities" of a diameter which does not meet the curve.

167. **Prop. VII.** *If* CP, CD *be conjugate radii, and* PN, DR *the ordinates of* P, D, *the triangles* PCN, DCR *will be equivalent.*



For $\angle PCN =$ complement of DCR [160.

$$= CDR.$$

Hence the triangles are similar. And their sides CP, CD are equal. Therefore they are equivalent.

168. **Cor.** Since $CN = DR$, and $CR = PN$,

therefore $CN^2 - CR^2 = DR^2 - PN^2 = CN^2 - PN^2$

$$= CA^2.$$ [142.

169. **Cor.** Also

$$DR^2 - CR^2 = CN^2 - PN^2 = CA^2,$$

where CD may be *any radius which does not meet the curve.*

---

[*] In the next figure a circle is drawn to indicate this equality. This must not be confounded with the auxiliary circle.

170. PROP. VIII. *The parallelogram* which has its sides equal and parallel to conjugate diameters is of constant area.*

In the last figure let CP, CD intersect in O.

Then since $\triangle DCR = PCN$, [167.

subtracting COR, $\triangle OCD = PORN$,

and adding POD, $\triangle PCD = PDRN$,

or $2\triangle PCD = (DR + PN)\,NR\dagger = (CN + CR)(CN - CR)$

$= CA^2$. [168.

For the rest compare Art. 77, and see fig. Art. 200.

171. PROP. IX. *The angle between two diameters is equal to that between their conjugates.*

For if CE be an asymptote, and CP, CD; Cp, Cd conjugate radii, such that

$\angle PCE = DCE$, [162.

and $\angle pCE = dCE$,

by subtraction $\angle PCp = DCd$.

172. More generally: The acute or obtuse angle between any two chords is equal to the acute or obtuse angle between any two chords conjugate to the former each to each. [70.

173. Since by Art. 71 supplemental chords as PQ, $P'Q$ and PQ', $P'Q'$ are conjugate, therefore the angles between PQ, PQ' are equal to those between $P'Q$, $P'Q'$. Hence any chord QQ' subtends at the ends of any diameter PP' angles which are either equal or supplementary.

It may be shewn that the angles subtended are equal or supplementary according as the *direction* of QQ' falls within or without PP'.

The next figure shews one of the four cases which occur.

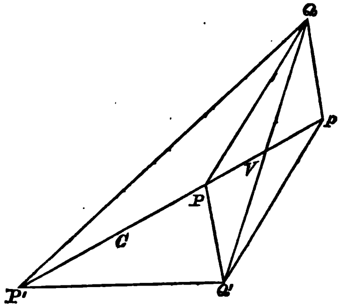
---

* In this case a rhombus.

† For $PDRN$ is made up of the triangles DRN, PND.

174. PROP. X. *If* CV *be the abscissa measured on the diameter* PCP′ *of a point* Q *on the curve, then*

$$QV^2 = PV \,.\, P'V = CV^2 - CP^2.$$



Produce QV to meet the curve in Q'. Then QQ' subtends at P, P' angles which, not being equal, are supplementary. [173.

On CV produced take Vp equal to VP. Then since QQ', Pp bisect one another they are the diagonals of a parallelogram.

Therefore $\angle QpQ' = QPQ' =$ supplement of $QP'Q'$.

Hence p, Q, P', Q' are concyclic, [Euc. III. 22.

and $QV^2 = pV \,.\, P'V$. [Euc. III. 35.

Therefore $QV^2 = PV \,.\, P'V = CV^2 - CP^2$.

175. Hence another proof of Prop. IV.:

On QQ' take VR, VR' each equal to CV.

Then $QR \,.\, QR' = RV^2 \sim QV^2 = CV^2 \sim QV^2$

$= CP^2$,

whence it may be shewn that CR, CR' are the asymptotes.

176. If Cv be the abscissa of Q measured parallel to QQ', then evidently Qv is equal to CV, and Cv to QV.

Therefore $Qv^2 - CP^2 = Cv^2$.

177. PROP. XI. *Diameters at right angles are equal.*

In the figure on p. 68 draw CP' at right angles to CD below the axis, and let CP be conjugate to CD.

Then $\angle DCA + PCA =$ a right angle [160.

$= P'CA + DCA$, by construction.

Hence the angles PCA, $P'CA$ are equal.

Therefore $CP' = CP = CD$. [146.

178. The ordinates in Prop. VII. have been taken to be principal ordinates. More generally, suppose CA to be *any* radius: then since the lines CP, CD and CN, DR are conjugate, therefore

$$\angle PCN = CDR. \qquad [172.$$

Hence the triangles PCN, CDR, having also the angles at C, N equal and the sides CP, CD equal, are equivalent.

The corollaries then follow with the help of Prop. X.

EXAMPLES.

179. In the rectangular hyperbola diameters at right angles bisect chords at right angles.

180. The bisectors of the angles between supplemental chords of a rectangular hyperbola are parallel to fixed lines.

181. Given a diameter and one other point of a rectangular hyperbola, construct the curve.

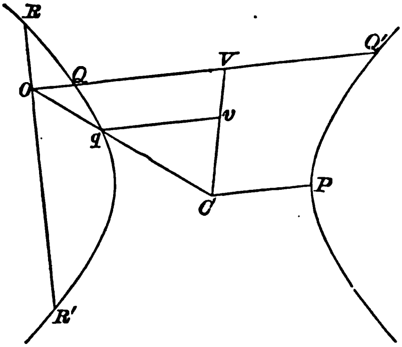
182. If on an arc AB of a circle whose centre is O there be taken points P, Q such that

$$\text{arc } AP = 2 \text{ arc } BQ,$$

then a rectangular hyperbola described on AO as diameter so as to pass through the intersection of OB with the tangent to the circle at A, will also pass through the intersection of AP, OQ.

183. Shew how to trisect a given circular arc by describing rectangular hyperbolas.

179. Prop. XII. *If a chord QQ′ passes through a fixed point O, the rectangle QO . OQ′ varies as the square of the parallel radius CP.*



Let V be the centre of the chord; Cq the radius towards O; Cv the abscissa of q.

Then $$QV^2 - CP^2 = CV^2,$$ [176.

and $$qv^2 - CP^2 = Cv^2.$$

Therefore

$$QV^2 - CP^2 : qv^2 - CP^2 = CV^2 : Cv^2$$
$$= OV^2 : qv^2, \text{ by parallels.}$$

Hence $OV^2 - QV^2 + CP^2 : CP^2 = OV^2 : qv^2$.

But the right-hand ratio is equal to $CO^2 : Cq^2$, which is constant since O, q are fixed points.

Therefore the left-hand ratio is constant.

Hence $OV^2 - QV^2 : CP^2$ is a constant ratio.

Therefore $QO . OQ'$ varies as CP^2.

180. The proof is similar when Q, Q' lie on the same branch. When CO does not meet the curve the above result may be obtained with the help of Art. 178.

181. Hence the rectangles contained by the segments of any two intersecting chords are as the squares of the parallel radii.

182. PROP. XIII. *If* PSQ *be the focal chord parallel to the radius* CD, *then*

$$SP \,.\, SQ = CD^2.$$

For, the axis itself being a focal chord,

$$SP \,.\, SQ : CD^2 = SA \,.\, SA' : CA^2, \qquad [181.$$

which is a ratio of equality. [141.

183. COR. Focal chords at right angles are equal. [177.

184. If QQ', RR' be chords at right angles, meeting in O, then

$$QO \,.\, OQ' = RO \,.\, OR'. \qquad [177.$$

The points Q, Q', R, R' are so situated that they cannot lie on one circle. Hence no exception to Art. 87 arises.

185. Since $QO : OR = OR' : OQ'$,

therefore QR' is perpendicular to $Q'R$, and QR to $Q'R'$. Hence, if any triangle $RQ'R'$ be inscribed in a rectangular hyperbola, its *Orthocentre* Q will lie on the curve.

186. Conversely, a conic which passes through the orthocentre and the vertices of a triangle must be a rectangular hyperbola; for it has three pairs of equal diameters at right angles, and it is evident that it cannot be a circle.

187. It may be shewn that the centre of a rectangular hyperbola described about a triangle lies on the nine-point circle, since the diameters of the hyperbola drawn to the middle points of the sides contain two and two the same angles as the sides. [172.

188. To prove again that Q in Art. 185 is on the curve: The straight line joining the middle point of RR' to V, the middle point of QQ', being a diameter of the nine-point circle, subtends a right angle at C. [187.

Hence CV bisects chords at right angles to RR'. Therefore Q is on the curve.

CHAPTER VI.

TANGENT-PROPERTIES OF THE RECTANGULAR HYPERBOLA.

189. In the figure on p. 62 suppose pP' to be any chord, and P such a point that the tangent thereat meets the chord at right angles in a point T. Then, since by a limiting case of Art. 184,

$$TP^2 = Tp \,.\, TP',$$

therefore Pp, PP' are at right angles. Hence the following construction for the tangent at P:

Draw any two chords Pp, PP′ *at right angles. Then the perpendicular from* P *on* pP′ *is the tangent required.*

190. The rectangular hyperbola has no tangents at right angles. For if the tangents from a point T be produced beyond the curve to meet the asymptotes in O, O', then OO', subtending a right angle at C, cannot subtend a right angle at T. [Euc. I. 21.

191. The portion Tt of any tangent intercepted by the asymptotes is bisected at the point of contact P. For by Art. 157, if a chord Qq meets the asymptotes in R, r the intercepts QR, qr are equal; and when the chord becomes the tangent at P these intercepts become PT, Pt. [Fig. Art. 200.

Hence $\qquad PT = Pt = CP = CD.$ [166.

192. Hence the tangents at the ends of a diameter P, P' meet the asymptotes at the vertices of a rhombus $TT't't$, which is of constant area $4CA^2$. [170.

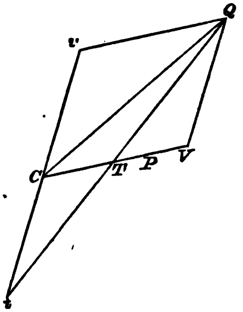
193. Also $QR \,.\, QR'$ becomes equal to PT^2 or CD^2.

194. Art. 172 includes the case of tangents regarded as limiting cases of bisected chords. Thus, in the next figure, QT being the tangent at Q,

since CQ, QT are conjugate,
and also CV, QV,
therefore $\angle\, QCV = TQV = QtC$.

195. PROP. XIV. *If* CV, QV *be the coordinates of a point* Q *on the curve, and if the tangent at* Q *meet the radius of abscissæ* CP *in* T, *then*

$$CV \,.\, CT = CP^2.$$



For since $\angle\, QCV = TQV$, [194.
the triangles QCV, TQV, equiangular at V, are similar, so that

$$CV : QV = QV : VT,$$
or
$$CV \,.\, VT = QV^2.$$
Hence
$$CV \,.\, CT = CV^2 - QV^2$$
$$= CP^2. \qquad [174.$$

196. When the radius of abscissæ does not meet the curve the points V, T assume positions v, t on opposite sides of C. The same method applies.

197. If the tangent at Q meets any two conjugate diameters in T, t, the rectangle $QT \,.\, Qt$ is equal to CQ^2.

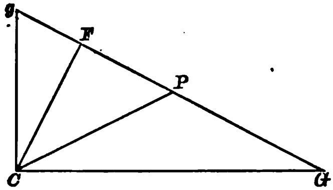
For the triangles QCT, QtC have their angles at Q, Q; C, t; T, C equal. [194.

198. PROP. XV. *If the normal at* P *meets the axes in* G, g, *then*

$$PG = CP = Pg.$$

The normal cuts the diameter conjugate to CP at right angles. Hence, F being the point of section,

$$\angle PGC = \text{complement of } FCG = PCG. \qquad [160.$$

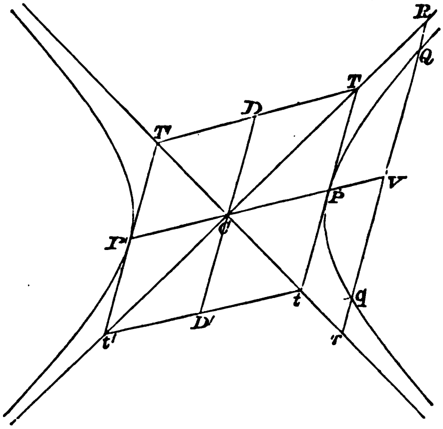


Therefore $PG = CP = Pg$, similarly.

199. COR. Draw perpendiculars GK, gk to SP.

Then $Pk = PK = CA$. [121, 140.

200. PROP. XVI. *Any tangent contains with the asymptotes a triangle of constant area.*



For if the tangent at P meet the asymptotes in T, t, and CD be conjugate to CP, then since the triangles CPT, CPD are between the same parallels, and since P bisects Tt, [191.

$$\triangle\, TCt = 2CPT = 2CPD$$

$$= CA^2. \qquad [170.$$

EXAMPLES.

184. The subnormal is equal to the principal abscissa.

185. The circle on SS' meets the asymptotes on the tangents at A, A'.

186. The circle through C which has its centre at P on the curve passes through the points in which the tangent and normal at P meet the asymptotes and the axes respectively.

187. Draw a tangent to a rectangular hyperbola which shall be parallel to a given line.

188. What is the locus of the middle point of a line which cuts off a constant area from the corner of a square?

189. If two concentric rectangular hyperbolas be such that the axes of one are the asymptotes of the other, they will intersect at right angles.

190. If two concentric rectangular hyperbolas touch the same straight line, the lines joining their points of intersection to their respective points of contact subtend equal angles at the centre.

191. If a right-angled triangle be inscribed in a rectangular hyperbola, the hypotenuse will be parallel to the normal at the opposite angle.

192. If two rectangular hyperbolas touch one another their common chords through the point of contact will include a right angle, and the remaining common chord will be parallel to their common tangent.

193. A circle through the centre and two points of a rectangular hyperbola passes also through the intersection of the lines drawn from each of the two points parallel to the polar of the other.

CHAPTER VII.

CHORD-PROPERTIES OF THE HYPERBOLA.

In dealing with the Hyperbola generally we shall have not so much to investigate a new set of propositions as to observe the particulars in which the curve under consideration differs from the ellipse, or, as regards asymptote-properties, from the rectangular hyperbola.

201. Let S be the focus of the hyperbola, and X the point in which the axis meets the directrix. The vertices A, A' satisfy the relation

$$SA : AX = SA' : A'X,$$

each of these ratios being equal to the eccentricity. Since the eccentricity is greater than unity it is evident that A, A' lie on opposite sides of X.

From the above proportion we deduce that

$$SA + SA' : AX + A'X = SA \sim SA' : AX \sim A'X,$$

each of these ratios being also equal to $SA : AX$.

Hence, if C be the middle point of AA',

$$\begin{aligned} 2CS : 2CA &= 2CA : 2CX \\ &= SA : AX. \end{aligned}$$

202. With the same construction as for the ellipse it may be shewn that in fig. 2

$$\angle ZSZ' = \text{a right angle}, \qquad [49.$$

and that $PN^2 : AN . NA' = SX^2 : AX . A'X$, [53.
which is a constant ratio.

203. The square of the ordinate varies as $NA \,.\, NA'$, or as the square of the tangent from N to the Auxiliary Circle. [144.

204. If we define CB as that value of PN which results from making AN and $A'N$ in the above proportion each equal to CA, then we may write, analogously to Art. 54,

$$PN^2 : AN \,.\, NA' = CB^2 : CA^2,$$

or

$$PN^2 : CN^2 - CA^2 = CB^2 : CA^2;$$

but in this case B *is not a point on the curve.*

205. The method of Art. 145 may now be applied *mutatis mutandis* to determine the form of the curve; the main difference being that the lines CE, CE' are to be taken subject to the relation

$$EN^2 : CN^2 = CB^2 : CA^2;$$

whence it appears that, as before, PN is less than EN, and the curve lies wholly within the angle ECE' and that vertically opposite.

206. DEF. The hyperbola may be called *Acute* or *Obtuse* according as the angle ECE' is acute or obtuse; or, in other words, according as CB is less or greater than CA.

DEF. On a perpendicular through C to AA' take equal lengths CB, CB'. The lines AA', BB' are called the *Transverse Axis* and the *Conjugate Axis* respectively. The former is also called the *Major Axis*, although it may be less than BB', and the latter is called the *Minor Axis.*

207. Through A draw a perpendicular to the axis, and let it meet the lines CE, CE' in H, h. Then from the construction in Art. 205,

$$AH = CB = Ah, \qquad \text{[Fig. 1.}$$

and

$$CH^2 = CA^2 + CB^2.$$

208. As in the first and second parts respectively of Art. 67, it may be shewn that if $2L$ be the latus rectum of the hyperbola

$$L \cdot CA = CB^2,$$

and $$L \cdot CA = CS \cdot SX.$$

Hence $$CB^2 = CS \cdot SX,$$

and $$CA^2 + CB^2 = CS \cdot CX + CS \cdot SX \qquad [201.$$
$$= CS^2.$$

Also $$SA \cdot SA' = CS^2 - CA^2$$
$$= CB^2.$$

209. Using the method of Art. 68 and the figure on p. 66, we may deduce from Art. 204 that

$$OL \cdot DR : CL \cdot CR = CB^2 : CA^2.$$

Hence, if CP, CD be conjugate radii, and CN, CR the abscissæ of P, D, then

$$PN \cdot DR : CN \cdot CR = CB^2 : CA^2. \qquad [70.$$

Or we may apply Arts. 94, 5.

210. DEF. The rectangular hyperbola described on AA' as axis may be called the *Auxiliary Hyperbola.* A diameter of this curve may be called, with reference to an acute or obtuse hyperbola on the same axis, an *Auxiliary Diameter.*

211. If the ordinate NP of any point on the hyperbola meet the auxiliary hyperbola in p, then, as in Art. 72,

$$PN : pN = CB : CA.$$

212. *To draw conjugate diameters.*

Take auxiliary conjugate radii Cp, Cd, and let the ordinates pN, dR meet the hyperbola in P, D. [Fig. 3.

Then $$PN \cdot DR : pN \cdot dR = CB^2 : CA^2. \qquad [211.$$

Hence, the triangles pCN, dCR being equivalent,

$$PN \cdot DR : CR \cdot CN = CB^2 : CA^2. \qquad [167, 8.$$

Therefore CP, CD are conjugate. [209.

213. Def. If Cd does not meet the auxiliary curve, CD also will not meet its curve. But we may define CD as terminated by the ordinate of d.

214. Conversely, if CP, CD be conjugate,

$$PN : CR = CB : CA = DR : CN. \qquad [75.$$

215. Hence $DR^2 - PN^2 : CB^2 = CN^2 - CR^2 : CA^2$.

But $\qquad CN^2 - CR^2 = CA^2; \qquad [168.$

therefore $\qquad DR^2 - PN^2 = CB^2$.

Hence $CN^2 + PN^2 - CR^2 - DR^2 = CA^2 - CB^2$,

or $\qquad CP^2 \quad - \quad CD^2 \quad = CA^2 - CB^2;$

that is to say, *the difference of the squares of conjugate diameters of a hyperbola is constant.*

216. As in Art. 77, the rectilinear areas [Fig. 3.

$$PCN,\ DCR,\ PDRN$$
$$pCN,\ dCR,\ pdRN,$$

are each to each in the ratio of CB to CA.

But $\qquad \triangle PCD = PDRN + DCR - PCN,$

and $\qquad \triangle pCd = pdRN + dCR - pCN;$

therefore $\qquad \triangle PCD$ is to pCd, that is $\frac{1}{2}CA^2$, $\qquad$ [170.

as CB to CA.

Hence the triangle PCD is equal to $\frac{1}{2}CA \cdot CB$, and the parallelogram which has its sides equal and parallel to conjugate diameters is equal to $AA' \cdot BB'$. [Fig. 1.

217. Art. 84 applies to the hyperbola, if instead of Euc. III. 35, 6 Art. 179 be assumed.

218. Also $QV^2 : PV \cdot VP' = CD^2 : CP^2$. $\qquad$ [89.

Hence $\qquad QV^2 : CV^2 - CP^2 = CD^2 : CP^2$,

and $\qquad QV^2 + CD^2 : CV^2 = CD^2 : CP^2$.

219. Analogously to Art. 204, CD^2 is the value of QV^2 which would result from equating PV and $P'V$ to CP.

THE ASYMPTOTES.

220. Through P (fig. Art. 200) draw a parallel to QV, and take PT, Pt each equal to CD. Let CT, Ct cut QV in R, r.

Then $$RV^2 : CV^2 = PT^2 : CP^2$$
$$= CD^2 : CP^2.$$

Hence $$RV^2 = QV^2 + CD^2.$$ [218.

Therefore $$RQ \,.\, Qr = RV^2 - QV^2$$
$$= CD^2.$$

And since RQ diminishes indefinitely as the ordinate recedes from the axis, therefore CR, or Cr, is an asymptote. [165.

221. From this construction it appears that if a straight line meets the curve in Q, q and the asymptotes in R, r, then since the same diameter bisects Qq and Rr, the intercepts QR, qr are equal.

Also it appears that as Qq moves parallel to itself, the rectangle $RQ \,.\, Qr$ has the constant value CD^2.

This constant value becomes CB^2 when Qq is parallel to the conjugate axis, and CA^2 when it is parallel to the transverse axis.

222. If from any point Q on the curve, Qx, Qy be drawn each parallel to one asymptote and terminated by the other, then Rr being supposed to move parallel to itself, one of the lines Qx, Qy, say Qx, varies as QR, and the other Qy varies as Qr. But $RQ \,.\, Qr$ is constant. Therefore $Qx \,.\, Qy$ is constant. [220.

When Q is at A, Qx and Qy become $\frac{1}{2}\,Ch$ and $\frac{1}{2}\,CH$. [207.

Hence always $$Qx \,.\, Qy = \tfrac{1}{4}\,(CA^2 + CB^2)$$
$$= \tfrac{1}{4}\,CS^2.$$ [208.

CHAPTER VIII.

TANGENT-PROPERTIES OF THE HYPERBOLA.

223. FROM the construction of Art. 220, it follows that the portion Tt of any tangent between the asymptotes is bisected at the point of contact. [Fig. Art. 200.

224. The triangle TCt is of constant area, being equal to $2CPD$, or $CA \cdot CB$. [200, 216.

225. The triangles TCt, HCh being equal, their sides about the angle C are proportional, so that

$$CT : CH = Ch : Ct.$$ [Fig. 1.

Therefore $CT \cdot Ct = CH \cdot Ch = CS^2$. [207, 8.

226. Art. 103 applies to the hyperbola when CP meets the curve*.

If V, T assume positions v, t on the conjugate diameter DD', which does not meet the curve, then

$$Ct : QV = CT : VT.$$ [Fig. p. 75.

Hence, Cv being equal to QV,

$$\begin{aligned} Cv \cdot Ct : QV^2 &= CV \cdot CT : CV \cdot VT \\ &= CP^2 : CV^2 - CP^2, \end{aligned}$$

by the first case.

Therefore $Cv \cdot Ct = CD^2$. [218.

Fig 5 suits the axis-property in an acute hyperbola.

Proof: $CN : CA = CO : CR$

$= CA : CT.$

Therefore $CN \cdot CT = CA^2$.

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* Arts. 195, 6 may be extended by Projection.

227. In Art. 110, it is true in all cases that

$$\angle TSL = TSM.$$

When the tangents are drawn as in fig. 4, to *opposite branches* of a hyperbola, the one angle is supplementary to that which TP subtends at S, and the other is equal to that which TQ subtends at the same point.

Hence TP, TQ *subtend supplementary angles at* S.

In like manner it may be shewn that the angle $TS'P$ is equal to the supplement of the angle $TS'Q$.

228. The tangent at P bisects the angle SPS'. [Fig. 4.

The proof in Art. 113 is applicable.

Or we may proceed as follows in the case of either curve:

Since PR subtends right angles at M, S, the points P, M, R, S are concyclic, and [105.

$$\angle SPR = SMX.$$

Similarly $\angle S'PR' = S'M'X'$.

Hence, the triangles SMX, $S'M'X'$ being equivalent,

$$\angle SPR = S'PR'.$$

229. The normal at P, being at right angles to PR, is equally inclined to the focal distances SP, $S'P$.

230. Art. 114 applies to the hyperbola, except that Pg is in this case the direction of the tangent, and Pt that of the normal. If a confocal ellipse and hyperbola intersect in P, their tangents at P will be at right angles.

231. If TP, TP' touch opposite branches of a hyperbola it may be shewn that

$$\angle STP = S'TP'. \qquad [115.$$

If the tangents be drawn to the same branch these angles may be shewn to be supplementary.

232. Arts. 122—8 apply *mutatis mutandis* to the hyperbola.

Let SY, $S'P$ meet in s. Then it may be shewn that

$$2CY = S's = S'P - SP \qquad \text{[Fig. 8.}$$

$$= 2CA. \qquad \text{[147.}$$

Hence the locus of Y, or Y', is the auxiliary circle. [138.

Let Z be the point diametrically opposite to Y' in which SY produced meets the circle.

Then since evidently $SZ = S'Y'$,

$$SY \, . \, S'Y' = SY \, . \, SZ = SA \, . \, SA'$$

$$= CB^2. \qquad \text{[208.}$$

233. It may be shewn that in an acute hyperbola tangents at right angles intersect on a fixed circle. For if T be the point, between Y, Y', in which two such tangents intersect*, then

$$CA^2 - CT^2 = TY \, . \, TY' \qquad \text{[Euc. III. 35.}$$

$$= CB^2, \text{ as in Art. 129.}$$

234. The method of Art. 131 applies to the hyperbola. [Fig. 7.

235. Art. 135 applies to the hyperbola. [Fig. 6.

236. So too does the preceding Article.

237. Various minor Articles in preceding chapters apply with or without modification, as the case may be, to the general hyperbola; and many Examples too which have been stated for the ellipse only, apply also to the hyperbola. The student should make a practice of considering in every case, including that of the parabola, whether a theorem is applicable *mutatis mutandis* to more than one of the three species of Conics.

EXAMPLES.

194. The centre of an equilateral hyperbola described about an equilateral triangle is on the inscribed circle.

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* The curve in fig. 8 is not drawn to suit this case.

195. In the rectangular hyperbola SPS' is a right angle when $CP^2 = 2CA^2$.

196. A focal perpendicular on an asymptote of a rectangular hyperbola is equal to the semi-axis. What is the corresponding property of the general hyperbola?

197. CY being drawn perpendicular to the tangent at P, the triangles PCA, CAY are similar.

198. The intersections of any two conjugate chords of a rectangular hyperbola with the asymptotes cannot be concyclic.

199. A chord of a rectangular hyperbola meets the asymptotes in R, r and passes through a fixed point O. Shew that $RO \, . \, Or$ varies as the square of the parallel diameter.

200. A rectangular hyperbola confocal with an ellipse cuts it at the ends of the equal conjugate diameters.

201. If CP, CD be conjugate radii of a rectangular hyperbola, then will D be the reflexion of P with respect to one of the asymptotes.

202. Ellipses being inscribed in a parallelogram, their foci lie on a rectangular hyperbola.

203. If lines be drawn from any point on a rectangular hyperbola to the ends of a diameter the difference of the angles which they make with the diameter will be equal to the angle which it makes with its conjugate.

204. From fixed points A, B straight lines are drawn intersecting in a point C such that the difference of the angles CBA, CAB is constant. Find the locus of C.

205. A conic through the four common points of two rectangular hyperbolas is itself a rectangular hyperbola.

206. A conic through the centres of the four circles which touch the sides of a triangle is a rectangular hyperbola, and its centre is on the circumscribing circle.

207. Any chord of a rectangular hyperbola subtends equal or supplementary angles at the ends of a perpendicular chord.

208. On opposite sides of a chord of a rectangular hyperbola equal segments of circles are described. Shew that the four points in which the circles meet the curve again are the vertices of a parallelogram.

209. The tangents to a rectangular hyperbola at the vertices of an inscribed triangle meet two and two on the lines joining the feet of the perpendiculars of the triangle.

210. If each vertex of a triangle be the pole of the opposite side with respect to an equilateral hyperbola, the circumscribing circle will pass through the centre of the hyperbola.

211. Straight lines joining the ends of conjugate focal chords meet on the asymptotes.

212. A circle and a rectangular hyperbola intersect in four points. If one of their common chords is a diameter of the hyperbola, the other is a diameter of the circle.

213. A parallelogram $ABCD$ has its diagonal AC at right angles to the side AB. If CD be divided into any number of equal parts and straight lines be drawn from A to the points of section, and if AC be divided into the same number of equal parts and straight lines be drawn from B to the points of section, then will corresponding lines in the two series meet on a hyperbola.

214. Hence shew how to construct a rectangular hyperbola.

215. If N be a point in AA' produced the circles described about S, S' with radii AN, $A'N$ meet on the hyperbola. What is the corresponding construction for the ellipse?

216. The difference of the focal distances of any point is greater or less than the axis according as the point lies on the concave or the convex side of the hyperbola.

217. If SY be perpendicular to the tangent at P

$$SY^2 : CB^2 = SP : 2CA + SP.$$

218. From a fixed point O, OP is drawn to a given circle. Find the envelope of a straight line through P inclined at a constant angle to OP.

219. The tangent from N to the circle on XX' varies as the normal.

220. In a central conic a circle through P and either G or g cuts off from the focal distances lengths whose sum is constant.

221. Given in an ellipse a focus and two points, the other focus describes a hyperbola.

222. If P, Q be points on a central conic a confocal passes through the intersections of SP, $S'Q$ and SQ, $S'P$.

223. The tangents at these points and at P, Q cointersect.

224. Given the asymptotes and a point on the curve, find the foci.

225. In the hyperbola SP is equal to a line drawn from P parallel to an asymptote to meet the directrix.

226. If two hyperbolas have the same asymptotes a chord of one touching the other is bisected at the point of contact.

227. The straight lines joining the points in which two tangents meet the asymptotes are parallel.

228. The common chords of a circle and a hyperbola make equal angles with the asymptotes.

229. The second tangents to a central conic from the points in which a tangent meets conjugate diameters are parallel.

230. If the tangent at P meets conjugate diameters in T, t, the triangles SPT, $S'Pt$ will be similar.

231. A hyperbola can be drawn through the ends of any two radii of an ellipse so as to have the conjugate diameters as asymptotes.

232. If PP', DD' be conjugate diameters of a hyperbola and Q any point on the curve, shew that $QP^2 + QP'^2$ exceeds $QD^2 + QD'^2$ by a constant quantity.

233. Given two points of a parabola and the direction of its axis, the locus of the focus is a hyperbola.

234. The centre of the inscribed circle of the triangle SPS' lies on the tangent at the nearer vertex.

235. A chord which subtends a right angle at the vertex meets the axis in a fixed point.

236. Find the locus of a point which divides the part of any tangent between the asymptotes in a constant ratio.

237. If a hyperbola touches the sides of a quadrilateral inscribed in a circle, and if one focus lies on the circle, the other lies on the circle.

CHAPTER IX.

ORTHOGONAL PROJECTION.

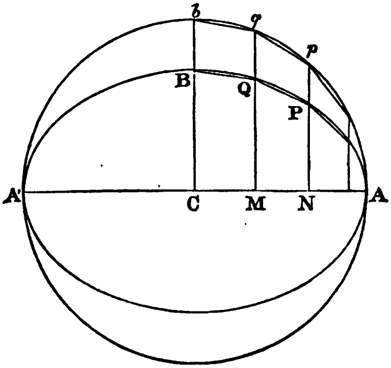
238. We have already made use of a system of *Auxiliary Points* in proving propositions which relate to the ratios of parallel straight lines. [84, 217.

The same method has been applied, though not expressly, to the evaluation of areas. [77, 216.

We shall first evaluate by this method the area of the ellipse, and then proceed to explain *Orthogonal Projection*, which is a method of obtaining the same results as those obtained by *Auxiliary Points*.

239. Prop. I. *The area of an ellipse is to that of its auxiliary circle as* CB *to* CA.

Take on the elliptic quadrant *AB* any number of points



as P, Q,... and let p, q,... be their auxiliary points. Draw the ordinates pPM, qQN,...

Since each ordinate is cut in the ratio $CB : CA$ by the ellipse and the circle, the area of each rectilinear figure $PQMN$ is to its auxiliary figure $pqMN$ as CB to CA, and the sum of the first series is to that of the second in the same ratio. [Euc. v. 12.

Hence, whatever be the number of points P, Q,... the rectilinear areas $CBPQ...A$, $Cbpq...A$ are as CB to CA.

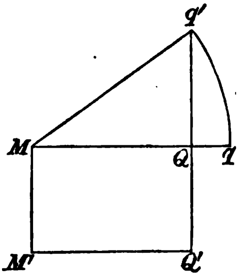
When the number of points is increased and the interval between successive points decreased indefinitely, the former of these two areas becomes the *elliptic* area $CBPQ...A$, and the latter becomes the *circular* area $Cbpq...A$.

Hence the corresponding elliptic and circular areas are as CB to CA*.

240. The results deducible by such methods as the foregoing depend of course upon the essential analogy between the ellipse and the circle, and not merely upon accident of relative position, or upon the magnitude of the particular circle. Thus the circle on BB' serves equally well as an auxiliary circle. [91.

Or, again, we may proceed as follows:

Suppose the ellipse to remain fixed in the plane of the



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* Hence the area of the ellipse is $\pi \,.\, CA \,.\, CB$.

paper, while the circle turns about its diameter AA'. Consider any auxiliary point q. This moves, in a plane through MQ perpendicular to the paper, into a new position q' at the same distance from M. Thus, instead of a straight line MQq cut in a constant ratio, we have a triangle MQq' whose sides MQ, Mq' are in the same constant ratio.

When the angle of rotation at M has reached a certain magnitude, the auxiliary point q' will be vertically over Q.

The angle at M is that through which the plane of the circle has turned, and is the same for all auxiliary points. Hence when q' is vertically over Q, the new position of every other auxiliary point as p will be vertically above its point P.

Now suppose the ellipse to descend vertically below the paper, without any oblique or angular displacement.

Then every point as Q assumes a new position Q' in $q'Q$ produced; corresponding lines as $M'Q'$, Mq' are in the original constant ratio $CB : CA$; and the new system of points Q', M',... are Orthogonal Projections of q', M,... according to the following

DEFINITION.

241. The *Orthogonal Projection* upon a plane of any point is the foot of the perpendicular let fall from the point upon the plane; and the projection of a locus is the curve or area which contains the projections of all points of the locus.

The line in which the plane of a figure intersects the plane of its projection may be called the *Axis of Projection.*

242. Through a straight line Mq' draw a plane perpendicular to that of projection; and let the common section of the planes be $M'Q'$. This line contains the projection of all points on Mq'. Hence ***straight lines project into straight lines, and conversely.***

243. It is evident that lengths parallel to the axis of projection are unaltered by projection; and also that every straight line meets its projection upon the axis in a point which is its own projection.

244. The segments of any given straight line $PQR...$ bear, each to each, the same ratio to the segments of its projection $Pqr...$ [Euc. VI. 2.

In particular, *points of bisection project into points of bisection.*

245. Through any number of parallels draw planes perpendicular to that of projection. Their common sections will be parallels. Hence *parallels project into parallels.*

246. Hence, and by Art. 244, bisected parallel chords project into bisected parallel chords. Therefore *ordinates project into ordinates.*

247. Project parallels $PQR...$, $P'Q'R'...$ into parallels $Pqr...$, $P'q'r'...$.

All the segments of the first line project in the ratio $PQ : Pq$; and all those of the second in the ratio $P'Q' : P'q'$. [244.

But by similar triangles these two ratios are equal.

Hence, alternando, *the ratios of the segments of parallels are unaltered by projection,* and ratios formed by compounding the ratios of the segments of parallels are unaltered by projection.

248. Let A, B be straight lines intersecting in O; a, b, o projections of A, B, O. Then because O lies on A, its projection o lies on a. Similarly, because O lies on B, o lies on b. That is to say, o is the intersection of a, b. Hence *points of intersection project into points of intersection.*

Hence if A passes through a fixed point O, then a passes through a fixed point o.

249. Project a chord PQ into a chord pq.

Let Q coalesce with P; then q coalesces with p. Hence *a tangent at any point P projects into a tangent at p.*

250. *Areas project in the same constant ratio as principal ordinates.* That is ordinates perpendicular to the axis of projection.

Let A, B, C,... be portions of principal ordinates intercepted by the bounding line or lines of the area. And first suppose the area rectilinear.

Then since A, B, C,..., being parallel, project in a constant ratio, and since lengths parallel to the axis are unaltered by projection, the portion of the area between any two of the lines as A, B projects in the same constant ratio.

Hence the whole rectilinear area projects in the same ratio as principal ordinates. [Euc. V. 12.

This may be extended to curvilinear areas, as in Art. 239.

251. PROP. II. *To project an ellipse into a circle, and conversely, to project a circle into an ellipse.*

(i) In the ellipse, PN being an ordinate of the *minor axis*,

$$PN^2 : BN . NB' = CA^2 : CB^2.$$

Take the axis of projection parallel to BB', and the angle between PN and its projection* pn (which is the same for all ordinates), such that

$$PN^2 : pn^2 = CA^2 : CB^2.$$

Hence
$$pn^2 = BN . NB'$$
$$= bn . nb',$$

since lengths parallel to the axis project unaltered. [243.

The locus of p is therefore a circle on a diameter equal to the minor axis of the ellipse. Another method is indicated in Chapter X.

(ii) The converse problem is indeterminate. For in the circle,

$$PN^2 = AN . NA',$$

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* The angle between the planes must be $\cos^{-1} \frac{CB}{CA}$.

where PN is an ordinate of the diameter AA'. And PN projects on any given plane in a *constant ratio* of less inequality. [247.

Now suppose AA' parallel to the axis of projection.

Then in the projected figure, pn^2 is to $AN.NA'$, or $an.na'$, in a constant ratio of less inequality, or the locus of p is an ellipse, whose *major axis* is equal to AA'.

252. In like manner a hyperbola may be projected into a rectangular hyperbola on the same axis. [211.

253. A parabola may be projected into a parabola. [Ex. 1.

254. We have seen that if q be the "Auxiliary Point" of Q, its position relatively to Q may be so altered, according to a certain law, that Q will be its projection.

By reversing this process we may bring a system of points into the same plane with their projections, so that the former become the "Auxiliary Points" of the latter.

The method of Auxiliary Points is in fact a method of *Plane Projection.* In the following applications the term *Projection* may be understood indifferently of either method.

255. *The locus of the middle points of parallel chords of an ellipse is a straight line.* [68.

Project the ellipse into a circle. Then the parallel chords project into parallel chords of the circle. [245.

The middle points of the former system of chords project into the middle points of the latter. [244.

But in the circle parallel chords are bisected by a straight line. Hence the locus of the middle points of any system of parallel chords of the ellipse projects into a straight line, and is therefore itself a straight line. [242.

256. Hence, *conjugate diameters of an ellipse project* into

conjugate diameters of a circle, that is to say *into diameters at right angles in a circle.* [75.

Conversely, diameters at right angles in a circle project into conjugate diameters of an ellipse.

257. *To find the relation between the oblique co-ordinates of an ellipse,*

$$QV^2 : PV . VP' = CD^2 : CP^2.$$

In the figure on p. 44 supply the radius CD.

Project the ellipse into a circle. Then the ordinate QV projects into an ordinate of the circle. [246.

And $QV^2 : CD^2$, being a ratio of parallels, projects unaltered. [247.

Similarly the ratio $PV . VP' : CP^2$ projects unaltered.

The above proportion will therefore hold for the ellipse if it holds for the circle.

Now, if the figure were taken to represent a circle, we should have $QV^2 = PV . VP'$; and $CD^2 = CP^2$. Hence the proportion holds for the circle. Therefore also in the ellipse,

$$QV^2 : PV . VP' = CD^2 : CP^2.$$

258. The proof in Art. 84 is similar.

259. *To prove that in an ellipse*

$$CV . CT = CP^2,$$

QV *being an ordinate of* CP, *and* T *the point in which its diameter intersects the tangent at* Q.

Project the ellipse into a circle. Then the ordinate QV projects into an ordinate of the circle, and the tangent at Q into a tangent. [246, 9.

Also the point of intersection T projects into a corresponding point of intersection. [248.

Hence the ratio $CV \,.\, CT : CP^2$ projects unaltered. [244.

We have therefore to shew that in the circle this is a ratio of equality.

Now, taking the same figure to represent a circle, we see that $CV \,.\, CT = CQ^2$, by similar right-angled triangles. And $CQ^2 = CP^2$. Therefore $CV \,.\, CT : CP^2$ is a ratio of equality.

Therefore also in the ellipse

$$CV \,.\, CT = CP^2.$$

260. *The area of a parallelogram which circumscribes an ellipse and has its sides parallel to conjugate diameters is constant.* [99.

Project the ellipse into a circle. Then conjugate diameters project into diameters at right angles, and parallels into parallels. [256, 245.

Also tangents project into tangents. [249.

Hence the projection of the parallelogram is a square circumscribing the circle, and is therefore of constant area.

Therefore the parallelogram is constant, since any area whose projection is constant must itself be constant. [250.

261. The same method applies to the hyperbola. See note, p. 83.

262. If the tangent at p to a circle meets in t, t' any two diameters at right angles, then $pt \,.\, pt' = cp^2$. To what property of the ellipse does this correspond?

Let the theorem be written in the form $pt \,.\, pt' = cd^2$, where cd is the radius parallel to the tangent.

Project the circle into an ellipse. [251.

The diameters at right angles project into conjugate diameters. [256.

The tangent and cd project into a tangent TPT', and the parallel radius CD; and the ratio $pt \,.\, pt' : cd^2$ projects unaltered. [249, 7.

Hence in the ellipse, if the tangent at P meets conjugate diameters in T, T', then, CD being the radius parallel to the tangent,

$$PT.PT' = CD^2.$$

This example is intended to indicate a way of proceeding in order to *discover* properties of the ellipse by generalizing known theorems in the circle.

EQUILATERAL CONICS.

263. Some properties may be stated and proved in the same way for the circle and the rectangular hyperbola, as for example the theorems of Arts. 167*, 170—3. In Art. 174 it might have been deduced from the similarity of the triangles QVP, $P'VQ$ that $QV^2 = PV.P'V$; and the same may be proved in the same way for the circle. The like may be said of Arts. 195—7.

If now a common name, as *Equilateral Conic*, be used to designate the circle and the rectangular hyperbola, we may in proving analogous properties use a single enunciation applicable to all central conics. For example, with the notation of Art. 197, we have in the case of either of the equilateral conics,

$$QT.Qt = CQ^2, \text{ by similar triangles,}$$
$$= CD^2,$$

if CD be the radius parallel to Tt.

Hence, by orthogonal projection, the theorem is proved at once for *any central conic*, and not merely for the circle, as in the preceding Article.

To take another example, if Cp, Cd be conjugate radii of an equilateral conic, the triangles CpN, CRd are equal in all respects. [72, 167.

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* See note, p. vi.

Hence $$CN^2 \pm CR^2 = CA^2,$$
and $$pN^2 \pm dR^2 = CA^2.$$

It follows by projection that if CP, CD be conjugate radii of any central conic, then

$$CN^2 \pm CR^2 = CA^2,$$
and $$PN^2 \pm DR^2 = CB^2.$$

By addition, $$CP^2 \pm CD^2 = CA^2 \sim CB^2.$$

These examples will suffice to shew the advantage of an acquaintance with the rectangular hyperbola, as enabling us to prove the common properties of central conics by a uniform method.

EXAMPLES.

238. Can two ellipses be projected at once into circles?

239. What parallelogram, and what triangles, circumscribing an ellipse has the least area?

240. A polygon circumscribing an ellipse has its sides bisected at the curve. Prove that its area is constant; and find the locus of its vertices.

241. In the circle, a chord of constant length cuts off a constant area; envelopes a concentric circle; and is bisected at the point of contact. Are these properties projective?

242. If a chord of an ellipse passes through a fixed point, its pole lies on a fixed straight line.

243. The perpendiculars of a triangle inscribed in a circle cointersect. Express this in a form suitable for projection.

244. Parallels through the ends of conjugate diameters cut the ellipse again at the ends of conjugate diameters.

245. From a point O on an ellipse draw a chord OP equally inclined with the tangent at P to the axis; shewing that three solutions are possible.

CHAPTER X.

THE RIGHT CIRCULAR CONE.

264. EUCLID defines a "*Cone*" as a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed. The fixed side is called the *Axis* and the fixed angular point the *Vertex* of the Cone. It is evident that the sections of this cone by planes perpendicular to the axis are circles. Hence the surface above defined is called the *Right Circular Cone,* when it is required to distinguish it from other cones.

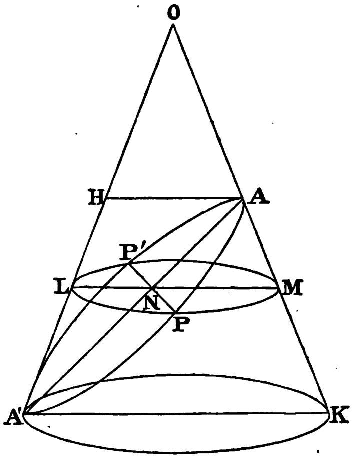
Euclid's definition gives only a finite portion of this cone. To obtain a complete cone we must produce the slant side of the triangle indefinitely both ways, so that it may generate an infinite surface divided symmetrically by the vertex. [Fig. p. 103.

Any straight line passing through the vertex and lying on the surface of the cone is called a *Generating* line. A plane through the axis cuts the surface in two generating lines, the constant angle between which is called the *Vertical Angle* of the cone.

265. If a generating line cut two given circular sections in Q, P, then, O being the vertex, OQ, OP are constant for all positions of the generating line. Hence $OQ \pm OP$ is constant. Therefore the intercept PQ is constant. [Fig. p. 101.

266. Let the plane of the paper contain the axis of the cone and the generating lines OL, OM.

Let *LPMP'* be a circular section, and *APA'P'* a section by any other plane perpendicular to that of the paper. Let



ANA' be the axis of the latter section, *N* being the point in which it cuts the diameter *LM* of the circle.

Then *PP'* is perpendicular to *LM* and *AA'*, and is bisected in *N*.

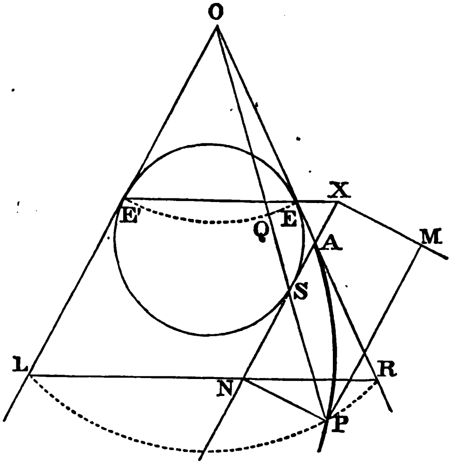
Hence $PN^2 = LN . NM$. [Euc. III. 31.

267. A Conic Section is the curve of intersection of a plane with a cone, and will in general be a Parabola, an Ellipse, or a Hyperbola.

In the following figure the lines which fall below the plane of the paper are not represented.

268. Prop. I. *To determine the nature of a given section of a right circular cone.*

Let *OE'L*, *OEAR* be two generating lines lying in the plane of the paper, which is taken perpendicular to *ANP* the plane of section.



In the cone inscribe a sphere touching the plane of the section in S, and let the plane of contact EQE' cut the plane of section in the line MX.

Let LPR be a circular section; PN the ordinate of P; PM perpendicular to MX; Q the point in which OP touches the sphere.

Then since PS, PQ are tangents to the same sphere,

$$SP = PQ = RE. \qquad [265.$$

Therefore $SP : NX = RE : NX$.

Hence $SP : PM = AE : AX$, by parallels,

$$= AS : AX,$$

since AE, AS, being tangents to the same sphere, are equal.

The locus of P is therefore a conic, having S for focus; MX for directrix; and $AE : AX$ for eccentricity.

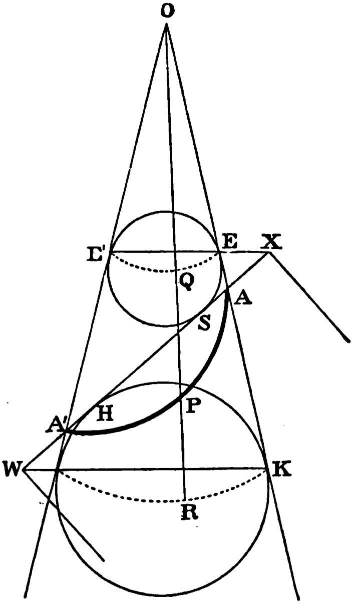
269. The eccentricity is equal to, less or greater than unity according as AE is equal to, less or greater than AX; or according as

$\angle$ AXE is equal to, less or greater than AEX.

But, the latter angle remaining constant, the angle AXE decreases or increases with the exterior angle EAN. Hence the section will be

(i) a *Parabola*, when its axis AN is parallel to a generating line as OL;

(ii) an *Ellipse*, when the angle EAN decreases, so that the extremities A, A' of the axis fall on the same side of the vertex;



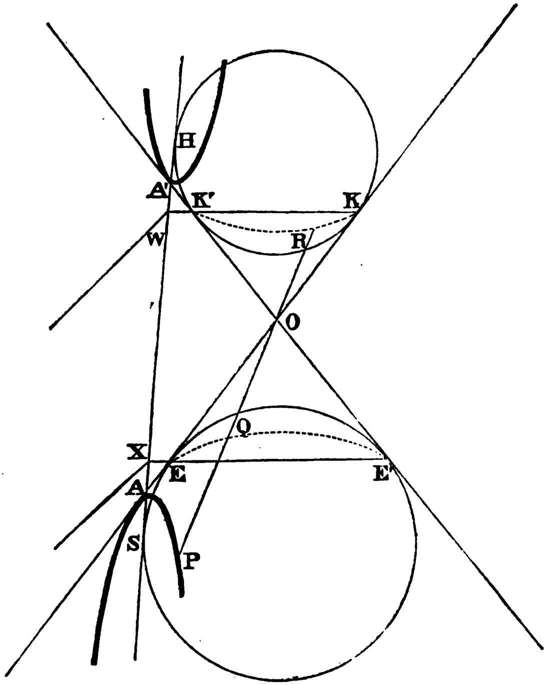
(iii) a *Hyperbola*, when the angle EAN increases, so that A, A' fall on opposite sides of the vertex.

270. Def. A sphere inscribed in a cone so as to touch a plane of section may be called a *Focal Sphere* of that section.

An elliptic section has two focal spheres on opposite sides of its plane.

A hyperbola has two focal spheres on the same side of its plane.

In the annexed figures S, H denote the foci; and lines are drawn through X, W to represent the directrices, these being the intersections of the plane of the conic with the planes of contact of the cone and the spheres.



The student will have no difficulty in shewing that

$$A'H = AS;$$

and that $$SP \pm HP = QR = AA'.$$ [265.

271. PROP. II. *At any point of a plane section the tangent makes equal angles with the focal distance and the generating line.*

In the preceding figures draw TP touching the section at P. Then, OPT being a tangent plane to the cone, TQ touches the focal sphere.

Now, by the equality of tangents from the same point to a sphere,

$$TS, PS = TQ, PQ, \text{ each to each.}$$

Hence the triangles PST, PQT, having the side PT common, are equal in all respects, so that

$$\angle SPT = QPT.$$

272. COR. Produce OP, TP, if necessary, to R, t; and let H be the second focus.

Then $$\angle HPt = RPt = QPT$$ [Euc. I. 15.
$$= SPT,$$

or *the tangent makes equal angles with SP, HP.* [113, 228.

273. PROP. III. *The semi-minor-axis of a section is a mean proportional to the radii of its focal spheres, and to the perpendiculars from the vertices of the section upon the axis of the cone.*

(i) Let AA' be the axis of a section; S, S' its foci; F, F' the centres of its focal spheres.

Then by a property of tangents to a circle, FA, $F'A$ bisect the angles between AA' and the generating line through A, and are therefore at right angles.

Hence, by similar right-angled triangles,

$$AS : FS = F'S' : AS';$$

therefore $$FS . F'S' = AS . AS' = CB^2,$$ [61, 208.

or CB is a mean proportional to the radii FS, $F'S'$.

(ii) As in Art. 266, any ordinate PN is a mean proportional to LN, NM; and these, when N bisects AA', become equal to $\frac{1}{2}AH$, $\frac{1}{2}A'K$ respectively, or to the perpendiculars from A, A' on the axis of the cone.

274. It is easy to deduce the theorems of Arts. 3, 53, 202 from the cone.

For PN^2, or $NM \,.\, LN$, varies as $AN \,.\, NA'$; [Fig. p. 100. or, if LN be supposed constant, PN^2 varies as NR, and therefore as AN. [Fig. p. 101.

275. *The latus rectum of a section varies as the perpendicular from the vertex of the cone upon the plane of section.*

In fig. p. 102 it is easily seen that OK is equal to the semi-perimeter of the triangle OAA'; and hence that, r, r' being the radii of its inscribed and escribed circles, and p the perpendicular from O on AA',

$$r \,.\, OK = \Delta OAA' = p \,.\, CA.$$

But OK varies as r', the angle $\frac{1}{2}AOA'$ being constant.

Hence $p \,.\, CA$ varies as $r \,.\, r'$, that is CB^2. [273.

Hence the latus rectum varies as p.

The student should consider cases of the hyperbola and parabola.

THE CYLINDER.

276. Let the vertical angle of a cone be diminished indefinitely so that its generating lines become ultimately *parallel* instead of passing through a fixed point. The surface which they generate is then said to be a *Right Circular Cylinder.*

The eccentricity of a plane section of the cylinder is $AE : EX$, with the notation of Art. 268; and it is easily seen that this expression may have any value from zero to unity.

Hence *an ellipse of any eccentricity* may be cut from a right circular cylinder, and *may* therefore *be projected orthogonally into a circle*, viz. upon a plane perpendicular to the generating lines.

APPENDIX.

The following duplicate proofs are worthy of consideration.

277. *Tangents to a conic subtend equal angles at* S.

It may be shewn from the equality of tangents to a circle, that a quadrilateral circumscribing a circle must have the sum or difference of two of its sides equal to that of the other two. Conversely, *if the sum or difference of two sides of a quadrilateral be equal to that of the other two, a circle can be inscribed in it.*

Take any two points P, P' on an ellipse. [115.

Then, because $SP + PS' = SP' + P'S'$, a circle touches the four sides of $SPS'P'$; and its centre will be at the intersection of the lines which bisect the exterior angles at P, P'; that is, at T the pole of PP'.

Now because SP, SP' touch the circle, and T is its centre, TS bisects the angle PSP', or TP, TP' *subtend equal angles at* S.

278. That *the tangent is equally inclined at* SP, $S'P$ may be shewn as follows:

(i) Let $xPpx'$ be a straight line cutting an ellipse in adjacent points P, p, the latter being the more remote from S. Draw perpendiculars PM, pm to Sp, $S'P$.

Now, since when p coincides with P the angles at S, S' vanish, we may consider that ultimately PSM, $pS'm$ are *isosceles* right-angled triangles, so that

$$Sp = SP + Mp, \text{ and } S'p + mP = S'P.$$

Hence $Mp = mP$, and, in the right-angled triangles PpM, pPm, the angles at p, P are equal.

The former is ultimately SPx. Hence the angles SPx, $S'Px'$ which the *tangent at* P makes with the focal distances are equal.

(ii) Conversely, in the figure of Art. 123, suppose PY equally inclined to SP, $S'P$. *Then will* PY' *be the tangent at* P.

Take any point Z on PY.

$$\text{Then} \qquad SZ + S'Z = SZ + sZ > Ss,$$
$$> AA'.$$

Hence every point of PY, *except* P, lies on the convex side of the curve; or PY *is the tangent at* P.

The same methods apply *mutatis mutandis* to the Hyperbola and to the Parabola.

279. *The tangent and normal at any point of a hyperbola meet the asymptotes and axes respectively in four points lying upon a circle which also passes through the centre of the hyperbola.*

The circle described on Gg, the portion of any normal intercepted by the axes, passes through the centre C, since gCG is a right angle.

Let this circle meet the asymptotes in L, M, and let LM, Gg intersect in P. From any point E in CL draw EN perpendicular to CG, and therefore parallel to gC.

Then $\angle ECN = GCM = GLP$, in the same segment.

Also $\angle CEN = ECg = LGP$, in the same segment.

Hence the triangles ECN, LPG are similar, so that the angle LPG *is a right angle*, and

$$PG : PL = EN : CN$$
$$= CB : CA. \qquad [205.$$

$$\text{Similarly} \qquad PL : Pg = CB : CA.$$

$$\text{Hence} \qquad PG : Pg = CB^2 : CA^2,$$

or P is the point at which G is normal. [119.

Also, LPG being a right angle, LM is the tangent at P.

280. Hence it is easily deduced:

(i) that the tangent LM is bisected at the curve; [223.

(ii) and that $\Delta LCM = PF \, . \, PL = CA \, . \, CB$, [235.

which is another form of the theorem of Art. 216.

281. CONJUGATE HYPERBOLAS

are such that the transverse axis of each is the conjugate axis of the other.

Conjugate hyperbolas have the same asymptotes, and are situated on opposite sides of them, as in Fig. 1.

282. In the rectangular hyperbola, radii at right angles are equal. [177.

Hence the locus of the extremity of the one will coincide, if turned through a right angle about the centre, with the locus of the extremity of the other.

Hence the extremities of diameters at right angles trace out *conjugate rectangular hyperbolas.*

283. It will serve as an exercise in orthogonal projection to generalize the following properties of the rectangular hyperbola. [Ex. 201.

The extremities of conjugate diameters trace out conjugate hyperbolas; the straight lines joining their extremities are parallel to the asymptotes; the tangents at their extremities meet on the asymptotes; the perpendiculars from their extremities to the axes meet two and two on the asymptotes.

CURVATURE.

284. From Art. 87 it follows readily by a *reductio ad absurdum*, that a circle cannot cut a conic in more points than four.

285. Let a circle cut a conic in P; and suppose a point, starting say from the concave side of the conic, to move round the circumference of the circle. At P the point passes to the convex side and cannot come round to its original position on the concave side without cutting the conic a second time.

Arguing in this way, we see that *a circle cannot cut a conic in an odd number of points.*

286. DEF. If a circle intersect a conic in four points P, P', Q, Q', and three of these points as P, P', Q coalesce, the circle is said to be the *Circle of Curvature* at P; and its chords through P are called *Chords of Curvature* of the conic.

287. The common chord PQ' of the circle and conic, and their common tangent, viz. at P, make equal angles with the axis. [87.

288. Except when P is on an axis of the conic, the circle of curvature will both touch and *cut* the conic at P, since it cannot cut it in one point Q' only. [285.

289. Let a circle cut a conic in Q, P, Q'. Draw any straight line through P so as to cut QQ', the circle, and the conic in V, R, P'. Let f, f' be focal chords of the conic parallel to QQ', PP'.

Then $QV.VQ' : PV.VP' = f : f'$. [88.

And $QV.VQ' = PV.VR$,

by a property of the circle.

Hence $VR : VP' = f : f'$.

Hence ultimately, when Q, V, Q' coalesce with P,

$$PR : PP' = f : f',$$

or at any point of a conic, the chords of curvature and of the conic in any direction are in the ratio of the focal chords parallel to the tangent and the chord of curvature.

290. The following are particular results deducible from the foregoing general theorem.

(i) The chord of curvature through S is equal to the focal chord of the conic drawn parallel to the tangent at P.

In a parabola, the chord of curvature through S, or parallel to the axis, is equal to $4SP$.

(ii) In a central conic, to determine the chord of curvature through C, we have

$$PR : PCP' = CD^2 : CP^2,$$

or $$PR = \frac{2CD^2}{CP}.$$

(iii) In a central conic, the radius of curvature is equal to $\dfrac{CD^3}{PF}$.

In any conic it is equal to $\dfrac{PG^3}{L^2}$.

(iv) At a point on a rectangular hyperbola, the diameter of curvature is equal to the normal chord of the conic; and the central chord of curvature is equal to the diameter of the conic; since $f = f'$ in each case.

THE BIFOCAL DEFINITION.

A central conic might have been defined as the locus of a point P, the sum or difference of whose distances from two fixed points S, H is constant. Taking the former case we may proceed as follows.

291. Describe the circle SPH, cutting the minor axis in g. Then Pg bisects the angle SPH, and is *the normal at P*. [278.

Let it cut the axis in G. It is easily seen that

$$SG : SP = HG : HP, \qquad \text{[Euc. VI. 3.}$$

then $$= CS : CA.$$

292. Produce gS, gH to meet a parallel to the axis in M, N. Draw perpendiculars MX, NW to the axis. We shall shew that these are *directrices*.

Since $\angle SMP = gSH = gHS$

$= gPS,$

therefore by similar triangles,

$$SP : PM = SG : SP$$

$$= CS : CA, \qquad [291.$$

or $SP : PM$ is a *constant ratio.* So too is $HP : PN$.

Again, from the foregoing proportions, we deduce that

$$SG : PM = CS^2 : CA^2,$$

whereof the former ratio is equal to $SH : MN$.

Therefore MN, or $2CX$, is constant, and MX is *a fixed straight line.* So too is NW.

293. The proofs are analogous when $SP \sim HP$ is constant.

EXAMPLES.

246. The three points on an ellipse whose circles of curvature pass through a given point on the curve are vertices of a triangle of which the centre of the ellipse is the intersection of the bisectors of the sides. [87, Ex. 245.

247. The normals at its vertices cointersect.

248. The latus rectum of a parabola cut from a given cone varies as the distance between the vertices of the section and the cone.

249. Shew that $SH = OA' - OA$. [Fig. p. 102.

250. The foci of similar elliptic sections of a cone lie on two cones.

251. The extremities of their minor axes lie on two generating lines.

252. Shew how to cut a right cone so that the axes of the section may be of given lengths.

253. If a section of a cone be projected on a plane perpendicular to the axis, a focus of the projection will lie on the axis of the cone.

254. Shew how to cut from a cone a section of given latus rectum.

255. A plane through the vertex parallel to that of a hyperbolic section cuts the cone in lines parallel to the asymptotes.

256. The latus rectum being constant, the envelope of the section is a sphere.

257. In a right-angled cone the perpendicular from the vertex upon a plane of section is equal to half the latus rectum.

258. Shew how to cut from a cone a hyperbola of maximum eccentricity.

259. The angle between the asymptotes being given, find the locus of the focus.

260. Under what circumstances is it possible to cut an equilateral hyperbola from a given cone?

261. The sphere on the line joining the centres of a pair of focal spheres contains the auxiliary circle of the section.

262. If Y, Y' be the feet of the perpendiculars from the foci of a section upon any tangent, and F, F' the centres of the focal spheres, shew that the triangles FSY, $F'S'Y'$ are similar, and that FY, YY', $F'Y'$ are mutually at right angles. [273.

263. Apply properties of the cone to prove the theorems of Articles 105, 110, 115, 122, 123.

264. Draw a tangent to a hyperbola which shall be parallel to a given straight line.

265. The polar with respect to a hyperbola of a point on the conjugate hyperbola touches the conjugate hyperbola.

266. Two parabolas have a common axis and vertex, and their concavities are opposed. The latus rectum of the one being eight times that of the other, prove that the portion of a tangent to the former intercepted between the axis and the tangent at the vertex is bisected by the latter.

267. If two focal chords of a rectangular hyperbola be drawn parallel to conjugate diameters, the straight lines joining their extremities will intersect on the asymptotes. Is this property projective?

268. If a point move in a plane so that the sum or difference of its distances from two fixed points, whereof one is in the plane, is constant; it will describe a plane section of a right cone whose vertex is at the other fixed point.

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